

# ON NON-ABELIAN QUADRIRATIONAL YANG-BAXTER MAPS

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*Dedicated to the Memory of  
Aristophanes Dimakis, 1953 – 2021*

ABSTRACT. We introduce four non-equivalent lists of families of non-abelian quadrirational Yang-Baxter maps, the so-called  $\mathcal{F}$ ,  $\mathcal{H}$ ,  $\mathcal{K}$  and  $\Lambda$  lists. We provide the canonical form of the generic map in each list, which under various degenerations lead to the remaining members of each list. In the abelian setting all four lists constitute the well known  $F$  and  $H$  lists of quadrirational Yang-Baxter maps.

*Keywords:* Yang-Baxter maps, non-abelian integrable difference systems

## 1. INTRODUCTION

In the recent years there is a growing interest in deriving and extending discrete and continuous integrable systems to the non-abelian domain [1, 2, 3, 4]. At the same time there is an intrinsic connection of discrete integrable systems with Yang-Baxter maps [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Although very important examples of non-commutative Yang-Baxter maps exist in the literature [16, 17, 18, 19, 20, 21, 22], the non-abelian counterparts of the Harrison map [6], a.k.a. the nonlinear superposition formula for the Bäcklund transformation of the Ernst equation [23], referred to as  $H_I$  in [24], and of the  $F_I$  [5] quadrirational Yang-Baxter maps are not known. The Harrison map  $H_I$  as well as the  $F_I$  map are the canonical forms of the generic maps (*top members*) of two non-equivalent lists of families of Yang-Baxter maps with five members each, the  $H$ -list and the  $F$ -list respectively. The maps  $H_I$  and  $F_I$  are considered top members of the corresponding lists since the remaining members can arise through degeneracies.

In this article we provide explicitly the non-abelian avatars of the  $H_I$  and the  $F_I$  Yang-Baxter maps, which participate as the generic maps of what we will call the  $\mathcal{H}$ -list and the  $\mathcal{F}$ -list respectively. In addition we provide the generic maps of two additional non-equivalent lists that we refer to as the  $\mathcal{K}$ -list and the  $\Lambda$ -list. The generic members of  $\mathcal{K}$ -list and the  $\Lambda$ -list in the abelian setting are both equivalent to the  $H_I$  Yang-Baxter map, which is not the case in the non-abelian setting as we show. Furthermore, all maps of the lists are naturally associated with integrable difference systems with variables defined on edges of an elementary cell of the  $\mathbb{Z}^2$  graph. We start this article with a short introduction followed by Section 2 where the basic definitions used throughout this paper are introduced. In addition, we provide the non-abelian version of the so-called Adler map as a motivating example. In Section 3, via a Lax pair formulation, we derive the non-abelian versions of the top members of the  $\mathcal{K}$ ,  $\Lambda$ ,  $\mathcal{H}$  and  $\mathcal{F}$  lists of families of

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quadrirational Yang-Baxter maps. In full extend these lists are presented in Appendix A. Finally, we conclude this article in Section 4 where we present some ideas for further research.

## 2. DEFINITIONS AND A NON-ABELIAN EXTENSION OF THE ADLER MAP

Let  $\mathbb{X}$  be any set. We proceed with the following definitions.

**Definition 1.** *The maps  $R : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X}$  and  $\widehat{R} : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X}$  will be called *YB equivalent* if it exists a bijection  $\kappa : \mathbb{X} \rightarrow \mathbb{X}$  such that  $(\kappa \times \kappa) R = \widehat{R} (\kappa \times \kappa)$ .*

**Definition 2.** *A bijection  $\phi : \mathbb{X} \rightarrow \mathbb{X}$  will be called *symmetry of the map  $R : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X}$ , if  $(\phi \times \phi) R = R (\phi \times \phi)$ .**

**Definition 3** (Yang-Baxter map). *A map  $R : \mathbb{X} \times \mathbb{X} \ni (u, v) \mapsto (x, y) = (x(u, v), y(u, v)) \in \mathbb{X} \times \mathbb{X}$ , will be called *Yang-Baxter map if it satisfies the Yang-Baxter relation**

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12},$$

where  $R_{i,j}$   $i, j \in \{1, 2, 3\}$ , denotes the action of the map  $R$  on the  $i$ -th and the  $j$ -th factor of  $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ , i.e.  $R_{12} : (u, v, w) \mapsto (x(u, v), y(u, v), w)$ ,  $R_{13} : (u, v, w) \mapsto (x(u, w), v, z(u, w))$ , and  $R_{23} : (u, v, w) \mapsto (u, y(v, w), z(v, w))$ .

Alternatively we can use the definition of *3D-compatible maps* [5]. Let  $F : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{X} \times \mathbb{X}$ , be a map and  $F_{ij}$   $i < j \in \{1, 2, 3\}$ , be the maps that act as  $F$  on the  $i$ -th and  $j$ -th factor of  $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ . For this definition it is convenient to denote these maps in components as follows

$$F_{ij} : (u^i, u^j, u^k) \mapsto (u_j^i, u_i^j, u^k) = (u_j^i(u^i, u^j), u_i^j(u^i, u^j), u^k), \quad i \neq j \neq k \neq i \in \{1, 2, 3\}.$$

**Definition 4** (3D-compatible map [5]). *A map  $F : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X}$  will be called *3D-compatible map if it holds  $u_{jk}^i = u_{kj}^i$  i.e.**

$$u_j^i \left( u_k^i(u^i, u^k), u_k^j(u^j, u^k) \right) = u_k^i \left( u_j^i(u^i, u^j), u_j^k(u^k, u^j) \right), \quad i \neq j \neq k \neq i \in \{1, 2, 3\}.$$

*Remark 2.1.* *YB equivalency respects the Yang-Baxter property as well as 3D-compatibility.*

The following proposition was considered in [24].

**Proposition 2.2.** *Let  $\phi : \mathbb{X} \rightarrow \mathbb{X}$  a symmetry of the Yang-Baxter map  $R : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X}$ . Then the map*

$$\widehat{R} = (\phi^{-1} \times id) R (id \times \phi),$$

*is also a Yang-Baxter map.*

Clearly the Yang-Baxter maps  $R$  and  $\widehat{R}$  are not *YB equivalent*. Hence, given a Yang-Baxter map and a symmetry of this map, one can introduce another Yang-Baxter map not equivalent with the original. Note that the same holds true for *3D-compatible maps*.

**Definition 5** ([25, 5]). *A map  $R : \mathbb{X} \times \mathbb{X} \ni (u, v) \mapsto (x, y) \in \mathbb{X} \times \mathbb{X}$  will be called *quadrirational, if both the map  $R$  and the so called companion map  $cR : \mathbb{X} \times \mathbb{X} \ni (x, v) \mapsto (u, y) \in \mathbb{X} \times \mathbb{X}$ , are birational maps.**

The notion of *quadrirational maps* first appeared in [25] under the name *non-degenerate rational maps*. Non-degenerate rational maps were renamed *quadrirational maps* in [5].

*Remark 2.3.* The companion map of a quadrirational Yang-Baxter map is a 3D-compatible map. The converse also holds i.e. the companion map of a quadrirational 3D-compatible map is a Yang-Baxter map.

**Definition 6** ([26, 27]). The matrix  $L(x; \lambda)$

- (1) is called a Lax matrix of the Yang-Baxter map  $R : (u, v) \mapsto (x, y)$ , if the relation  $R(u, v) = (x, y)$  implies that  $L(u; \lambda)L(v; \lambda) = L(y; \lambda)L(x; \lambda)$  for all  $\lambda$ ;
- (2) is called a Lax matrix of the companion map  $cR : (x, v) \mapsto (u, y)$ , if the relation  $cR(x, v) = (u, y)$  implies that  $L(u; \lambda)L(v; \lambda) = L(y; \lambda)L(x; \lambda)$  for all  $\lambda$ .  $L(x; \lambda)$  is called a strong Lax matrix of  $cR$  if the converse also holds.

For the rest of the article we consider the set  $\mathbb{X}$  to be  $\mathbb{D} \times \mathbb{D}$ , where  $\mathbb{D}$  is a non-commutative division ring, i.e. an associative algebra with a multiplicative identity element denoted by 1 and every non-zero element  $x$  of  $\mathbb{D}$  has a unique multiplicative inverse denoted by  $x^{-1}$  s.t.  $xx^{-1} = x^{-1}x = 1$ .

**2.1. Non-abelian extension of the Adler map.** A prototypical example of a Yang-Baxter map on  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  is the so-called Adler map (or  $H_V$ ) that was introduced in [28]. The Adler map reads:

$$H_V : (u, v) \mapsto (x, y) = \left( v - \frac{p-q}{u+v}, u + \frac{p-q}{u+v} \right).$$

For the rest of this Section and in the Propositions that follow, we extend the Adler map and its companion on  $\mathbb{X} \times \mathbb{X}$ , where  $\mathbb{X} := \mathbb{D} \times \mathbb{D}$ .

**Proposition 2.4.** The map  $c\mathcal{H}_V : (x^1, x^2, v^1, v^2) \mapsto (u^1, u^2, y^1, y^2)$ , where

$$\begin{aligned} u^1 &= (v^1(v^1 - x^1) + v^2v^1 - x^2x^1)(x^1 - v^1)^{-1}, & u^2 &= (v^1 - x^1)x^2x^1((v^1 + x^2)x^1 - (v^1 + v^2)v^1)^{-1}, \\ y^1 &= (x^1(x^1 - x^1) + x^2x^1 - v^2v^1)(v^1 - x^1)^{-1}, & y^2 &= (x^1 - v^1)v^2v^1((x^1 + v^2)v^1 - (x^1 + x^2)x^1)^{-1}, \end{aligned}$$

- (1) has as symmetry the bijection  $\phi : (z^1, z^2) \mapsto (-z^1, -z^2)$ ;
- (2) has as strong Lax matrix the matrix

$$L(x^1, x^2; \lambda) = \begin{pmatrix} x^1 & (x^1 + x^2)x^1 - \lambda \\ 1 & x^1 \end{pmatrix}, \quad (1)$$

where we assume that the spectral parameter  $\lambda$  belongs to the center  $C(\mathbb{D})$  of the algebra  $\mathbb{D}$ , i.e. it commutes with every element of  $\mathbb{D}$ ;

- (3) it is a 3D-compatible map.

*Proof.* Is is easy to verify that  $(\phi \times \phi)c\mathcal{H}_V = c\mathcal{H}_V(\phi \times \phi)$  and that proves that the bijection  $\phi$  is a symmetry of  $c\mathcal{H}_V$ . As a consequence, the map  $(u^i, v^i, x^i, y^i) \mapsto (-u^i, -v^i, -x^i, -y^i)$ ,  $\forall i \in \{1, 2\}$ , leaves invariant the compatibility conditions (2)-(5).

Let us now prove item (2). From the Lax equation  $L(u^1, u^2; \lambda)L(v^1, v^2; \lambda) = L(y^1, y^2; \lambda)L(x^1, x^2; \lambda)$ , we obtain the following compatibility conditions

$$u^1 + v^1 = y^1 + x^1, \quad (2)$$

$$u^1 v^1 + (v^1 + v^2)v^1 = y^1 x^1 + (x^1 + x^2)x^1, \quad (3)$$

$$(u^1 + u^2)u^1 + u^1 v^1 = (y^1 + y^2)y^1 + y^1 x^1, \quad (4)$$

$$(u^1 + u^2)u^1 v^1 + u^1(v^1 + v^2)v^1 = (y^1 + y^2)y^1 x^1 + y^1(x^1 + x^2)x^1. \quad (5)$$

First note that the compatibility conditions (2)-(5) are symmetric under the interchange

$$(u^1, u^2, v^1, v^2) \leftrightarrow (y^1, y^2, x^1, x^2). \quad (6)$$

Equations (2), (3) are linear in  $u^1, y^1$ , and do not include  $u^2, y^2$ , so the latter can be easily solved for  $u^1, y^1$ . Specifically, by eliminating  $y^1$  from (2), (3), we obtain

$$u^1 = (v^1(v^1 - x^1) + v^2 v^1 - x^2 x^1) (x^1 - v^1)^{-1}. \quad (7)$$

Applying (6) to (7) we get

$$y^1 = (x^1(x^1 - v^1) + x^2 x^1 - v^2 v^1) (v^1 - x^1)^{-1}. \quad (8)$$

Substituting (7),(8) to (4),(5) and by solving them we obtain

$$y^2 = (x^1 - v^1)v^2 v^1 (x^1(v^1 - x^1) + v^2 v^1 - x^2 x^1)^{-1}, \quad (9)$$

$$u^2 = (v^1 - x^1)x^2 x^1 (v^1(x^1 - v^1) + x^2 x^1 - v^2 v^1)^{-1}. \quad (10)$$

Equations (7),(8), (9) and (10), coincide with the defining relations of the map  $c\mathcal{H}_V$  of the proposition. Moreover, (7),(8), (9) and (10) is the unique solution of the compatibility conditions (2)-(5), so the Lax matrix  $L(x^1, x^2; \lambda)$  is strong.

The  $3D$ -compatibility of  $c\mathcal{H}_V$  can be proven by direct computation. Alternatively, by using the fact that  $L(x^1, x^2; \lambda)$  is strong, from [27, 29] the  $3D$ -compatibility follows.  $\square$

In the commutative setting where all variables are considered elements of the center of the algebra  $\mathbb{D}$ , from the defining relations of  $c\mathcal{H}_V$  we obtain that

$$u^2 u^1 = x^2 x^1, \quad y^2 y^1 = v^2 v^1,$$

so the products  $x^2 x^1$  and  $v^2 v^1$  are invariants of  $c\mathcal{H}_V$ . Clearly the latter are no longer invariants of the map in the non-commutative setting. Nevertheless, if we assume that the products  $x^2 x^1$  and  $v^2 v^1$  belong to the center of the algebra  $\mathbb{D}$ , i.e. they commute with all elements of  $\mathbb{D}$ , then these products are invariants of the map. This assumption is referred to as *centrality assumption* and it was firstly introduced in [1, 20] where it played an essential role in obtaining the companion map of the so-called  *$N$ -periodic reduction of the KP map*.

From further on, when we refer to the centrality assumption for a map  $F : (z^1, z^2, w^1, w^2) \mapsto (\bar{z}^1, \bar{z}^2, \bar{w}^1, \bar{w}^2)$  we refer to

$$z^2 z^1 = p \in C(\mathbb{D}), \quad w^2 w^1 = q \in C(\mathbb{D}), \quad (11)$$

where  $C(\mathbb{D})$  the center of the algebra  $\mathbb{D}$ . Note that as a consequence of (11), we have the commutativity relations  $z^2 z^1 = z^1 z^2$ ,  $w^2 w^1 = w^1 w^2$ .

**Proposition 2.5.** *Under the centrality assumption, the map  $cH_V$  of Proposition 2.4 is quadrirational with companion map that reads  $\mathcal{H}_V : (u^1, u^2, v^1, v^2) \mapsto (x^1, x^2, y^1, y^2)$ , where*

$$\begin{aligned} x^1 &= (u^1 + v^1)^{-1} ((u^1 + v^1)v^1 + v^2v^1 - u^2u^1), & x^2 &= u^2u^1 ((u^1 + v^1)v^1 + v^2v^1 - u^2u^1)^{-1} (u^1 + v^1), \\ y^1 &= (u^1 + v^1)^{-1} ((u^1 + v^1)u^1 + u^2u^1 - v^2v^1), & y^2 &= v^2v^1 ((u^1 + v^1)u^1 + u^1u^1 - v^2v^1)^{-1} (u^1 + v^1). \end{aligned}$$

The map  $\mathcal{H}_V$  is a Yang-Baxter map.

*Proof.* Note that under the centrality assumption (11), from the map  $c\mathcal{H}_V$  of Proposition 2.4 we obtain

$$x^2x^1 = u^2u^1 = p \in C(\mathbb{D}), \quad y^2y^1 = v^2v^1 = q \in C(\mathbb{D}). \quad (12)$$

The first defining relation of  $c\mathcal{H}_V$  of Proposition 2.4 reads

$$u^1 = -v^1 + (v^2v^1 - x^2x^1)(x^1 - v^1)^{-1},$$

by using  $x^2x^1 = u^2u^1$  from (12), we can solve for  $x^1$  to obtain

$$x^1 = (u^1 + v^1)^{-1} ((u^1 + v^1)v^1 + v^2v^1 - u^2u^1), \quad (13)$$

namely the first defining relation of  $\mathcal{H}_V$  mapping. Now we can substitute (13) to the second defining relation of  $c\mathcal{H}_V$  and solve for  $x^2$  in terms of  $u^i, v^i, i = 1, 2$ , or equivalently from (3) of the compatibility conditions, by using again  $x^2x^1 = u^2u^1$ , we obtain the second defining relation of  $\mathcal{H}_V$  mapping, namely

$$x^2 = u^2u^1 ((u^1 + v^1)v^1 + v^2v^1 - u^2u^1)^{-1} (u^1 + v^1). \quad (14)$$

Now we substitute (13) and (14) to the third and fourth defining relation of  $c\mathcal{H}_V$  of Proposition 2.4, to obtain

$$y^1 = (u^1 + v^1)^{-1} ((u^1 + v^1)u^1 + u^2u^1 - v^2v^1), \quad (15)$$

$$y^2 = v^2v^1 ((u^1 + v^1)u^1 + u^1u^1 - v^2v^1)^{-1} (u^1 + v^1). \quad (16)$$

(13)-(16) constitute the defining relations of  $H_V$  mapping and that completes the first part of the proof.

The proof that  $\mathcal{H}_V$  is a Yang-Baxter map follows from the fact that it is the companion of a  $3D$ -compatible map. □

The  $\mathcal{H}_V$  map serves as the non-abelian form of the Adler map ( $H_V$ ). This is apparent since under the change of variables  $(u, p, v, q) = (u^1, u^1u^2, v^1, v^1v^2)$ ,  $\mathcal{H}_V$  reads

$$\mathcal{H}_V : (u, p, v, q) \mapsto (x, p, y, q),$$

where

$$x = v + (u + v)^{-1}(q - p), \quad y = u + (u + v)^{-1}(p - q),$$

that clearly coincides with the Adler map in the commutative case.

## 3. NON-ABELIAN EXTENSION OF QUADRIRATIONAL YANG-BAXTER MAPS

**Proposition 3.1.** *Provided the centrality assumptions (11), the map*

$$c\mathcal{K}_{a,b,c} : (x^1, x^2, v^1, v^2) \mapsto (u^1, u^2, y^1, y^2),$$

where

$$\begin{aligned} u^1 &= (b - cv^1)(x^2 - v^2)x^1 (x^1 - v^1)^{-1} (a - cv^2)^{-1}, \\ u^2 &= (a - cv^2)(x^1 - v^1)x^2 (x^2 - v^2)^{-1} (b - cv^1)^{-1}, \\ y^1 &= (b - cx^1)(x^2 - v^2)v^1 (x^1 - v^1)^{-1} (a - cx^2)^{-1}, \\ y^2 &= (a - cx^2)(x^1 - v^1)v^2 (x^2 - v^2)^{-1} (b - cx^1)^{-1}, \end{aligned}$$

with  $a, b, c \in C(\mathbb{D})$  and neither  $a, c$  nor  $b, c$  simultaneously zero,

(1) has as symmetries the bijections

$$\psi : (z^1, z^2) \mapsto \left(\frac{b}{a}z^2, \frac{a}{b}z^1\right), \quad (17)$$

$$\phi : (z^1, z^2) \mapsto \left(\frac{b}{a}(a - cz^2)z^1(cz^1 - b)^{-1}, \frac{a}{b}(b - cz^1)z^2(cz^2 - a)^{-1}\right); \quad (18)$$

(2) has as strong Lax matrix the matrix

$$L(x^1, x^2; \lambda) = \begin{pmatrix} ax^1 - cx^2x^1 & \lambda(b - cx^1) \\ a - cx^2 & bx^2 - cx^1x^2 \end{pmatrix}, \quad (19)$$

where the spectral parameter  $\lambda \in C(\mathbb{D})$ ;

(3) it is a 3D-compatible map;

(4) it is quadrirational and its companion map reads

$$\mathcal{K}_{a,b,c} : (u^1, u^2, v^1, v^2) \mapsto (x^1, x^2, y^1, y^2),$$

where

$$\begin{aligned} x^1 &= (au^1 + bv^2 - c(v^1v^2 + u^1v^2))^{-1} u^1 (av^1 + bu^2 - c(v^2v^1 + u^2v^1)), \\ x^2 &= (bu^2 + av^1 - c(v^2v^1 + u^2v^1))^{-1} u^2 (bv^2 + au^1 - c(v^1v^2 + u^1v^2)), \\ y^1 &= (au^1 + bv^2 - c(u^1u^2 + u^1v^2)) v^1 (bu^2 + av^1 - c(u^2u^1 + u^2v^1))^{-1}, \\ y^2 &= (bu^2 + av^1 - c(u^2u^1 + u^2v^1)) v^2 (au^1 + bv^2 - c(u^1u^2 + u^1v^2))^{-1}. \end{aligned}$$

(5) The map  $\mathcal{K}_{a,b,c}$  is a Yang-Baxter map.

*Proof.* First note that as a consequence of the centrality assumption (11), from the map  $c\mathcal{K}_{a,b,c}$  we obtain

$$x^2x^1 = u^2u^1 = p \in C(\mathbb{D}), \quad y^2y^1 = v^2v^1 = q \in C(\mathbb{D}). \quad (20)$$

One can verify that  $(\phi \times \phi) c\mathcal{K}_{a,b,c} = c\mathcal{K}_{a,b,c} (\phi \times \phi)$ , as well as  $(\psi \times \psi) c\mathcal{K}_{a,b,c} = c\mathcal{K}_{a,b,c} (\psi \times \psi)$  and that proves that the bijections  $\phi$  and  $\psi$  are a symmetries of  $c\mathcal{K}_{a,b,c}$ . It can also easily shown that  $\psi^2 = \phi^2 = id$ , so  $\psi^{-1} = \psi$ ,  $\phi^{-1} = \phi$ , provided (20) holds.

From the Lax equation  $L(u^1, u^2; \lambda)L(v^1, v^2; \lambda) = L(y^1, y^2; \lambda)L(x^1, x^2; \lambda)$ , we obtain the following compatibility conditions

$$(b - cu^1)(a - cv^2) = (b - cy^1)(a - cx^2), \quad (21)$$

$$(a - cu^2)(b - cv^1) = (a - cy^2)(b - cx^1), \quad (22)$$

$$(b - cu^1)u^2(b - cv^1)v^2 = (b - cy^1)y^2(b - cx^1)x^2, \quad (23)$$

$$(a - cu^2)u^1(a - cv^2)v^1 = (a - cy^2)y^1(a - cx^2)x^1, \quad (24)$$

$$(b - cu^1)u^2(a - cv^2) + (a - cu^2)(a - cv^2)v^1 = (b - cy^1)y^2(a - cx^2) + (a - cy^2)(a - cx^2)x^1, \quad (25)$$

$$(a - cu^2)u^1(b - cv^1) + (b - cu^1)(b - cv^1)v^2 = (a - cy^2)y^1(b - cx^1) + (b - cy^1)(b - cx^1)x^2. \quad (26)$$

The system of equations (21)-(24) determines uniquely  $u^i, y^i$  as functions of  $v^i, x^i$ ,  $i = 1, 2$  i.e. the defining relations of the map  $c\mathcal{K}_{a,b,c}$  of this Proposition. Then, substituting  $c\mathcal{K}_{a,b,c}$  into (25),(26), the latter are satisfied provided (20) holds. So  $c\mathcal{K}_{a,b,c}$  is uniquely determined by (21)-(26) and that proves that (19) serves as a strong Lax matrix of  $c\mathcal{K}_{a,b,c}$ .

Using [26], the proof that  $\mathcal{K}_{a,b,c}$  is a Yang-Baxter map, follows directly from its Lax representation and the fact that it can be written as the projective action

$$x^1 = [u^1]L(v^1, v^2, u^2u^1), \quad x^2 = [u^2]L(v^1, v^2, u^1u^2), \quad y^1 = L(u^1, u^2, v^2v^1)[v^1], \quad y^2 = L(u^1, u^2, v^2v^1)[v^2],$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [x] := (ax + b)(cx + d)^{-1}, \quad [x] \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (xc + d)^{-1}(xa + b).$$

Then the  $3D$ -compatibility of  $c\mathcal{K}_{a,b,c}$  follows, since it serves as the companion map of a Yang-Baxter map. □

The following remarks are in order.

- For  $a, b, c \geq 0$ , the map  $\mathcal{K}_{a,b,-c}$ , is a totally positive Yang-Baxter map.
- Under the change of variables  $(u, p, v, q) = (u^1, u^1u^2, v^1, v^1v^2)$ , the symmetries  $\phi, \psi$ , obtain the form:

$$\psi : (u, p) \mapsto \left(\frac{b}{a}pu^{-1}, p\right), \quad (27)$$

$$\phi : (u, p) \mapsto \left(\frac{b}{a}(au - cp)(cu - b)^{-1}, p\right), \quad (28)$$

also  $\mathcal{K}_{a,b,c}$  reads:

$$\mathcal{K}_{a,b,c} : (u, p, v, q) \mapsto (x, p, y, q), \quad (29)$$

where

$$\begin{aligned} x &= v(auv + bq - cq(u + v))^{-1}(auv + bp - c(qu + pv)), \\ y &= (auv + bq - c(pv + qu))(auv + bp - cp(u + v))^{-1}u. \end{aligned}$$

In the following Proposition, we use the symmetries  $\phi$  and  $\psi$ , in order to obtain three additional non  $YB$  equivalent families of Yang-Baxter maps.

**Proposition 3.2.** *Let  $\mathcal{K}_{a,b,c} : (u, p, v, q) \mapsto (x, p, y, q)$ , be the Yang-Baxter map given in (29), with symmetries the bijections  $\psi, \phi$  given in (27),(28). The maps  $\Lambda_{a,b,c} := (\psi^{-1} \times id) \mathcal{K}_{a,b,c} (id \times \psi)$ ,  $\mathcal{H}_{a,b,c} := (\phi^{-1} \times id) \mathcal{K}_{a,b,c} (id \times \phi)$ , and  $\mathcal{F}_{a,b,c} := (\psi^{-1} \circ \phi^{-1} \times id) \mathcal{K}_{a,b,c} (id \times \phi \circ \psi)$ , where*

$$\begin{aligned} x &= pv (ab(qu + pv) - cq(bp + auv))^{-1} (ab(u + v) - c(bq + auv)), \\ y &= q (ab(u + v) - c(bp + auv)) (ab(qu + pv) - cp(bq + auv))^{-1} u, \end{aligned} \quad (\Lambda_{a,b,c})$$

$$\begin{aligned} x &= \left( (auv - bq)(v - \frac{c}{a}q)^{-1} - (auv - bp)(v - \frac{b}{c})^{-1} \right)^{-1} \\ &\quad \left( p(auv - bq)(\frac{a}{c}v - q)^{-1} - (auv - bp)(\frac{c}{b}v - 1)^{-1} \right), \end{aligned} \quad (\mathcal{H}_{a,b,c})$$

$$\begin{aligned} y &= (a(ab - c^2q)uv + abc(q - p)v + bq(c^2p - ab)) \\ &\quad (a(ab - c^2p)uv + abc(p - q)u + bp(c^2q - ab))^{-1} u, \end{aligned}$$

$$\begin{aligned} x &= p (cp(u - v)(b - cv)^{-1} - a(qu - pv)(cq - av)^{-1})^{-1} \\ &\quad (b(u - v)(b - cv)^{-1} - c(qu - pv)(cq - av)^{-1}), \\ y &= q ((ab - c^2q)u + bc(q - p) + (c^2p - ab)v) \\ &\quad (q(ab - c^2p)u + ac(p - q)uv + p(c^2q - ab)v)^{-1} u, \end{aligned} \quad (\mathcal{F}_{a,b,c})$$

are non-abelian quadrirational Yang-Baxter maps.

*Proof.* The proof follows directly by applying Proposition 2.2 to the map  $\mathcal{K}_{a,b,c}$ .  $\square$

In the abelian setting and for generic  $a, b, c$ , it holds that  $\mathcal{H}_{a,b,c}$  is related to  $\mathcal{K}_{a,b,c}$  through the conjugation  $\chi : z \mapsto (1 - z)^{-1}$  followed by  $(p, q) \mapsto ((1 - p)^{-1}, (1 - q)^{-1})$  [24], i.e.  $\mathcal{H}_{a,b,c} = \chi^{-1} \times \chi^{-1} \mathcal{K}_{a,b,c} \chi \times \chi$ . Also,  $\Lambda_{a,b,c}$  is related to  $\mathcal{K}_{a,b,c}$  through the conjugation  $\omega : z \mapsto z^{-1}$  followed by  $(p, q) \mapsto (p^{-1}, q^{-1})$ . So in the commutative setting essentially we have two non-equivalent families of Yang-Baxter maps, the family  $\mathcal{H}_{a,b,c}$  and the family  $\mathcal{F}_{a,b,c}$  which coincide with the families  $H$  and  $F$  in [5, 24]. In the non-abelian setting, the families  $\mathcal{K}_{a,b,c}$ ,  $\Lambda_{a,b,c}$  and  $\mathcal{H}_{a,b,c}$  are no longer equivalent under conjugation. In the commutative diagram of Figure 1, we present the four families of quadrirational Yang-Baxter maps  $\mathcal{K}_{a,b,c}$ ,  $\Lambda_{a,b,c}$ ,  $\mathcal{H}_{a,b,c}$ ,  $\mathcal{F}_{a,b,c}$  and their interrelations in the abelian and the non-abelian setting respectively, for generic  $a, b$  and  $c$ .

As a final remark, note that for  $a, b, c \geq 0$ , the map  $\Lambda_{a,b,-c}$ , is a totally positive Yang-Baxter map. Furthermore, note that all four families of non-abelian maps introduced in this Section, are non-involutive maps.

#### 4. CONCLUSIONS

In this article we used Lax formulation to introduce four lists of non-abelian quadrirational Yang-Baxter maps, by providing the canonical forms of the generic maps that correspond to each of these lists, namely the maps  $\mathcal{F}_{1,1,1}$ ,  $\mathcal{H}_{1,1,1}$ ,  $\mathcal{K}_{1,1,1}$  and  $\Lambda_{1,1,1}$ . Various degenerations of these canonical maps led to the remaining members of the corresponding  $\mathcal{F}$ ,  $\mathcal{H}$ ,  $\mathcal{K}$  and  $\Lambda$  lists of non-abelian Yang-Baxter maps presented in Appendix A.



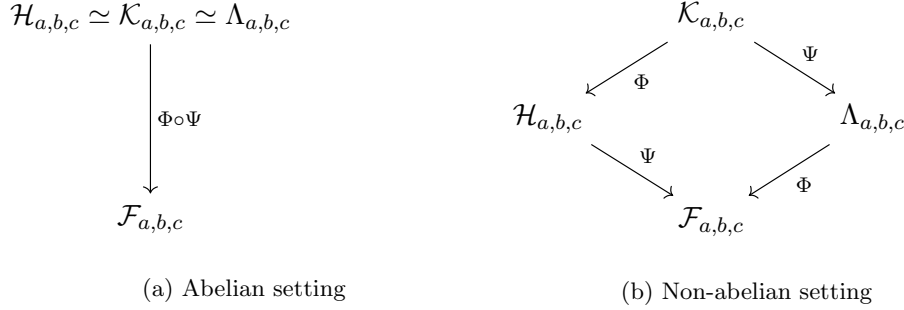


Figure 1. The four families of quadrirational Yang-Baxter maps in the abelian and in the non-abelian setting. The morphisms  $\Phi, \Psi$ , are respectively defined by  $\Phi : R \rightarrow (\phi^{-1} \times id)R(id \times \phi)$  and  $\Psi : R \rightarrow (\psi^{-1} \times id)R(id \times \psi)$ , where  $\phi, \psi$  the symmetries defined in (27),(28).

In the case of *entwining Yang-Baxter maps* [30], one can combine the results of this article together with the results of [31], to obtain non-abelian entwining Yang-Baxter maps associated with each member (apart  $\mathcal{F}_{IV}$ ) of the  $\mathcal{F}$ ,  $\mathcal{H}$ ,  $\mathcal{K}$  and  $\Lambda$  lists. Moreover, natural questions concerning the Liouville integrability of the transfer maps corresponding to these lists could be addressed. Furthermore, we anticipate the study of the corresponding to these lists integrable difference systems with vertex variables in a future work.

Finally note that the families of Yang-Baxter maps presented in this article serve as the lowest members of hierarchies of families of Yang-Baxter maps. For example the map  $\mathcal{K}_{a,b,c}$  is the companion map (for  $N=2$ ) of the following 3D-compatible hierarchy of maps [32]:

$$c\mathcal{K}_{a^1, \dots, a^N, c}^N : (x^1, \dots, x^N, v^1, \dots, v^N) \mapsto (u^1, \dots, u^N, y^1, \dots, y^N),$$

where

$$\begin{aligned} u^i &= (a^i - cv^i)(x^{i-1} - v^{i-1})x^i (x^i - v^i)^{-1} (a^{i-1} - cv^{i-1})^{-1}, \\ y^i &= (a^i - cx^i)(x^{i-1} - v^{i-1})v^i (x^i - v^i)^{-1} (a^{i-1} - cx^{i-1})^{-1}, \end{aligned} \quad i = 1, \dots, N,$$

with  $a^i, c \in C(\mathbb{D})$  and  $a^i, c \forall i \in \{1, \dots, N\}$  not simultaneously zero and the index  $i$  is considered modulo  $N$ . Note that

$$\psi_{a^1, \dots, a^N} : (z^1, \dots, z^i, \dots, z^N) \mapsto \left( \frac{a^1}{a^N} z^N, \dots, \frac{a^i}{a^i - 1} z^{i-1}, \dots, \frac{a^N}{a^{N-1}} z^{N-1} \right),$$

is a symmetry of  $c\mathcal{K}_{a^1, \dots, a^N, c}^N$ . The hierarchy  $c\mathcal{K}_{a^1, \dots, a^N, c}^N$  with  $c = 0$ , and  $a^i = 1, \forall i$ , i.e.  $c\mathcal{K}_{1, \dots, 1, 0}^N$ , coincides with the so-called  $N$ -periodic reduction of the KP map and it was firstly considered in [1, 20] c.f. [2]. Whereas the non-equivalent hierarchy  $(\psi_{1, \dots, 1}^{-1} \times id) c\mathcal{K}_{1, \dots, 1, 0}^N (id \times \psi_{1, \dots, 1})$  was firstly considered in [33]. Furthermore, since  $c\mathcal{K}_{1, \dots, 1, 0}^N$  with  $i \in \mathbb{Z}$  (or equivalently  $N \rightarrow \infty$ ) is the non-abelian KP map (Hirota-Miwa map), interesting and open questions concern the underlying geometry and the identification of  $c\mathcal{K}_{a^1, \dots, a^N, c}^N$  as an integrable difference system when  $i \in \mathbb{Z}$  and for generic  $a^1, \dots, a^N$  and  $c$ .

# Appendices

## APPENDIX A. THE NON-ABELIAN $\mathcal{F}, \mathcal{K}, \Lambda$ AND $\mathcal{H}$ LISTS

For generic  $a, b, c$ , the families of maps  $\mathcal{F}_{a,b,c}$ ,  $\mathcal{K}_{a,b,c}$ ,  $\Lambda_{a,b,c}$  and  $\mathcal{H}_{a,b,c}$ , given in Proposition 3.2 and in (29) are considered as the generic maps of the  $\mathcal{F}, \mathcal{K}, \Lambda$  and  $\mathcal{H}$  lists. For each generic map various degeneracies can occur by demanding for instance the coalescence of singularities of the generic map when restricted in the abelian domain. Using as a guiding principle this coalescence of singularities, in what follows we present the non-Abelian  $\mathcal{F}$  list. Then from the  $\mathcal{F}$ -list and by using Proposition 2.2 we obtain the  $\mathcal{K}, \Lambda$  and  $\mathcal{H}$  lists.

**The non-Abelian  $\mathcal{F}$ -list.** The non-Abelian  $\mathcal{F}$ -list of quadrirational Yang-Baxter maps reads:

$$R : (u, p, v, q) \mapsto (x, p, y, q)$$

where:

$$\begin{aligned} x &= p \left( p(u-v)(1-v)^{-1} - (qu-pv)(q-v)^{-1} \right)^{-1} \\ &\quad \left( (u-v)(1-v)^{-1} - (qu-pv)(q-v)^{-1} \right), \end{aligned} \quad (\mathcal{F}_I \equiv \mathcal{F}_{1,1,1})$$

$$y = q \left( (1-q)u + q - p + (p-1)v \right) \left( (1-p)u + (p-q)uv + p(q-1)v \right)^{-1} u,$$

$$\begin{aligned} x &= q^{-1}(1-v)(u-v)^{-1}(qu-pv+p-q)v(1-v)^{-1}, \\ y &= p^{-1}(qu-pv+p-q)(u-v)^{-1}u, \end{aligned} \quad (\mathcal{F}_{II} \equiv \mathcal{F}_{0,1,1})$$

$$\begin{aligned} x &= q^{-1}v(u-v)^{-1}(qu-pv), \\ y &= p^{-1}(qu-pv)(u-v)^{-1}u, \end{aligned} \quad (\mathcal{F}_{III} \equiv \mathcal{F}_{0,0,1})$$

$$\begin{aligned} x &= (u-v)^{-1}(u-v+p-q)v, \\ y &= (u-v+p-q)(u-v)^{-1}u, \end{aligned} \quad (\mathcal{F}_{IV})$$

$$\begin{aligned} x &= v + (p-q)(u-v)^{-1}, \\ y &= u + (p-q)(u-v)^{-1}, \end{aligned} \quad (\mathcal{F}_V)$$

Note that the  $\mathcal{F}_{IV}$  map is obtained from  $\mathcal{F}_{II}$  by setting

$$(x, y, u, v, p, q) \mapsto (1 + \epsilon x, 1 + \epsilon y, 1 + \epsilon u, 1 + \epsilon v, 1 + \epsilon p, 1 + \epsilon q)$$

and then sending  $\epsilon \rightarrow 0$ . Furthermore the  $\mathcal{F}_V$  map is obtained from  $\mathcal{F}_{IV}$  by setting

$$(x, y, u, v, p, q) \mapsto (1 + \epsilon x, 1 + \epsilon y, 1 + \epsilon u, 1 + \epsilon v, 1 + \epsilon^2 p, 1 + \epsilon^2 q)$$

and then sending  $\epsilon \rightarrow 0$ .

The non-abelian maps  $\mathcal{F}_{III}$  and  $\mathcal{F}_V$  were first introduced in [5].

**The non-Abelian  $\mathcal{K}$ -list.** The non-Abelian  $\mathcal{K}$ -list of quadrirational Yang-Baxter maps reads:

$$R : (u, p, v, q) \mapsto (x, p, y, q)$$

where:

$$\begin{aligned} x &= v(uv + q(1 - u - v))^{-1}(p + uv - qu - pv), \\ y &= (q + uv - qu + pv)(uv + p(1 - u - v))^{-1}u, \end{aligned} \quad (\mathcal{K}_I \equiv \mathcal{K}_{1,1,1})$$

$$\begin{aligned} x &= q^{-1}v(1 - u - v)^{-1}(p - qu - pv), \\ y &= p^{-1}(q - qu - pv)(1 - u - v)^{-1}u, \end{aligned} \quad (\mathcal{K}_{II} \equiv \mathcal{K}_{0,1,1})$$

$$\begin{aligned} x &= pv(qu + pv)^{-1}(u + v), \\ y &= q(u + v)(qu + pv)^{-1}u, \end{aligned} \quad (\mathcal{K}_{III})$$

The non-abelian map  $\mathcal{K}_{III}$  was first introduced in [33].

**The non-Abelian  $\Lambda$ -list.** The non-Abelian  $\Lambda$ -list of quadrirational Yang-Baxter maps reads:

$$R : (u, p, v, q) \mapsto (x, p, y, q)$$

where:

$$\begin{aligned} x &= pv(qu + pv - q(p + uv))^{-1}(u + v - q - uv), \\ y &= q(u + v - p - uv)(qu + pv - p(q + uv))^{-1}u, \end{aligned} \quad (\Lambda_I \equiv \Lambda_{1,1,1})$$

$$\begin{aligned} x &= pv(qu + pv - pq)^{-1}(u + v - q), \\ y &= q(u + v - p)(qu + pv - pq)^{-1}u, \end{aligned} \quad (\Lambda_{II})$$

$$\begin{aligned} x &= v(uv + q)^{-1}(uv + p), \\ y &= (uv + q)(uv + p)^{-1}u, \end{aligned} \quad (\Lambda_{III})$$

Note that  $\Lambda_{II}$  is obtained from  $(\Lambda_{a,b,c})$  by setting  $c \mapsto ab$  and then taking  $a = 0$ . We can take  $b = 0$  instead, but the map we obtain is equivalent up to conjugation with  $\chi(\alpha) : z \mapsto \alpha z^{-1}$ , with  $\Lambda_{II}$  i.e. this map reads  $(\chi(p) \times \chi(q)) \Lambda_{II} (\chi(p)^{-1} \times \chi(q)^{-1})$ .

The non-abelian map  $\Lambda_{III}$  was first introduced in [20].

**The non-Abelian  $\mathcal{H}$ -list.** The non-Abelian  $\mathcal{H}$ -list of quadrirational Yang-Baxter maps reads:

$$R : (u, p, v, q) \mapsto (x, p, y, q)$$

where:

$$\begin{aligned}
x &= v \left( (uv - q)(v - q)^{-1} - (uv - p)(v - 1)^{-1} \right)^{-1} \\
&\quad \left( (uv - q)(v - q)^{-1} - (uv - p)(v - 1)^{-1} \right), & (\mathcal{H}_I \equiv \mathcal{H}_{1,1,1}) \\
y &= ((1 - q)uv + (q - p)v + q(p - 1)) \left( (1 - p)uv + (p - q)u + p(q - 1) \right)^{-1} u,
\end{aligned}$$

$$\begin{aligned}
x &= v(uv - q)^{-1}(uv - p), \\
y &= (uv - q)(uv - p)^{-1}u, & (\mathcal{H}_{III})
\end{aligned}$$

$$\begin{aligned}
x &= v - (p - q)(u + v)^{-1}, \\
y &= u + (p - q)(u + v)^{-1}, & (\mathcal{H}_V)
\end{aligned}$$

$\mathcal{H}_V$  has the symmetry  $u \mapsto -u$ . Via this symmetry by using Proposition 2.2 we recover the  $\mathcal{F}_V$  map.

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