



On Finite Groups with an Automorphism of Prime Order Whose Fixed Points Have Bounded Engel Sinks

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Abstract

A left Engel sink of an element g of a group G is a set $\mathcal{L}(g)$ such that for every $x \in G$ all sufficiently long commutators $[\dots[[x, g], g], \dots, g]$ belong to $\mathcal{L}(g)$. (Thus, g is a left Engel element precisely when we can choose $\mathcal{L}(g) = \{1\}$.) We prove that if a finite group G admits an automorphism φ of prime order coprime to $|G|$ such that for some positive integer m every element of the centralizer $C_G(\varphi)$ has a left Engel sink of cardinality at most m , then the index of the second Fitting subgroup $F_2(G)$ is bounded in terms of m . A right Engel sink of an element g of a group G is a set $\mathcal{R}(g)$ such that for every $x \in G$ all sufficiently long commutators $[\dots[[g, x], x], \dots, x]$ belong to $\mathcal{R}(g)$. (Thus, g is a right Engel element precisely when we can choose $\mathcal{R}(g) = \{1\}$.) We prove that if a finite group G admits an automorphism φ of prime order coprime to $|G|$ such that for some positive integer m every element of the centralizer $C_G(\varphi)$ has a right Engel sink of cardinality at most m , then the index of the Fitting subgroup $F_1(G)$ is bounded in terms of m .

Keywords Finite groups · Engel condition · Fitting subgroup · Automorphism

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1 Introduction

We use the left-normed simple commutator notation

$$[a_1, a_2, a_3, \dots, a_r] := [\dots [[a_1, a_2], a_3], \dots, a_r]$$

and the abbreviation $[a, {}_k b] := [a, b, b, \dots, b]$ where b is repeated k times. Recall that an element g of a group G is called a left Engel element if for every $x \in G$ the equation $[x, {}_n g] = 1$ holds for some $n = n(x)$ depending on x . An element $g \in G$ is called a right Engel element if for every $x \in G$ the equation $[g, {}_n x] = 1$ holds for some $n = n(x)$. An Engel group is defined as a group in which every element is a left Engel element (equivalently, every element is a right Engel element). Finite Engel groups are nilpotent by Zorn's theorem (Robinson 1996, 12.3.4). Moreover, by Baer's theorem (Robinson 1996, 12.3.7), a left Engel element of a finite group belongs to its Fitting subgroup, and a right Engel element of a finite group belongs to its hypercentre. For many classes of groups theorems related to properties of Engel elements have been proved, achieving various structural properties related to nilpotency. For example, Wilson and Zelmanov (1992) proved that profinite Engel groups are locally nilpotent. At the same time, Golod's examples (Golod 1964) show that Engel groups in general may not be locally nilpotent.

In recent papers (Khukhro and Shumyatsky 2016, 2018a, b, 2019, 2020a; Khukhro et al. 2019) we considered generalizations of Engel conditions for finite, profinite, and compact groups using the concept of Engel sinks.

Definition A *left Engel sink* of an element g of a group G is a set $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x, g, g, \dots, g]$ belong to $\mathcal{E}(g)$, that is, for every $x \in G$ there is a positive integer $l(x, g)$ such that $[x, {}_l g] \in \mathcal{E}(g)$ for all $l \geq l(x, g)$.

(Thus, g is a left Engel element precisely when we can choose $\mathcal{E}(g) = \{1\}$, and G is an Engel group when we can choose $\mathcal{E}(g) = \{1\}$ for all $g \in G$.)

Definition A *right Engel sink* of an element g of a group G is a set $\mathcal{R}(g)$ such that for every $x \in G$ all sufficiently long commutators $[g, x, x, \dots, x]$ belong to $\mathcal{R}(g)$, that is, for every $x \in G$ there is a positive integer $r(x, g)$ such that $[x, {}_r g] \in \mathcal{R}(g)$ for all $r \geq r(x, g)$.

(Thus, g is a right Engel element precisely when we can choose $\mathcal{R}(g) = \{1\}$, and G is an Engel group when we can choose $\mathcal{R}(g) = \{1\}$ for all $g \in G$.)

In Khukhro and Shumyatsky (2018a, 2020a) we considered finite, profinite, and compact (Hausdorff) groups in which every element has a finite or countable left Engel sink and proved that then the group is close to being locally nilpotent in a certain precise sense. In Khukhro and Shumyatsky (2019, 2020b) we obtained similar results concerning right Engel sinks.

When G is a finite group, then every element has the *smallest* left Engel sink, since the intersection of two left Engel sinks $\mathcal{E}'(g)$ and $\mathcal{E}''(g)$ is again a left Engel sink of g . Similarly, every element $g \in G$ has the *smallest* right Engel sink. In this paper

we shall always use the notation $\mathcal{L}(g)$ and $\mathcal{R}(g)$ to denote the smallest left and right Engel sinks of g , respectively, thus eliminating the ambiguity of this notation in the above definitions.

For finite groups we proved in Khukhro and Shumyatsky (2018a, 2019) the following quantitative results.

Theorem 1.1 (Khukhro and Shumyatsky 2018a, Theorem 3.1) *Let G be a finite group, and m a positive integer. Suppose that every element $g \in G$ has a left Engel sink $\mathcal{L}(g)$ of cardinality at most m . Then G has a normal subgroup N of order bounded in terms of m such that G/N is nilpotent.*

Theorem 1.2 (Khukhro and Shumyatsky 2019, Theorem 3.1) *Let G be a finite group, and m a positive integer. Suppose that every element $g \in G$ has a right Engel sink $\mathcal{R}(g)$ of cardinality at most m . Then G has a normal subgroup N of order bounded in terms of m such that G/N is nilpotent.*

As is well known, often strong consequences for the structure of the group can be derived from certain conditions imposed on fixed points of automorphisms. Some of the best-known examples include Thompson's theorem (Thompson 1959) on the nilpotency of a finite group with a fixed-point-free automorphism of prime order, as well as numerous other papers on finite groups admitting automorphisms with various restrictions on their fixed points. This type of results in relation to automorphisms whose fixed points have restrictions on their Engel sinks were recently obtained in Acciarri et al. (2018, 2019b) Acciarri et al. (2019a) Acciarri and Silveira (2018, 2020), Khukhro and Shumyatsky (2020c).

The purpose of this paper is to obtain the following 'automorphism extensions' of Theorems 1.1 and 1.2; one may also regard these extensions as results on 'almost fixed-point-free' automorphisms.

Theorem 1.3 *Let G be a finite group admitting an automorphism φ of prime order coprime to $|G|$. Let m be a positive integer such that every element $g \in C_G(\varphi)$ has a left Engel sink $\mathcal{L}(g)$ of cardinality at most m . Then G has a metanilpotent normal subgroup of index bounded in terms of m only.*

The conclusion for right Engel sinks is stronger. Note that while it is well-known that the inverse of a right Engel element is left Engel, there is no such a straightforward connection between left and right Engel sinks, and Theorem 1.4 is not a consequence of Theorem 1.3.

Theorem 1.4 *Let G be a finite group admitting an automorphism φ of prime order coprime to $|G|$. Let m be a positive integer such that every element $g \in C_G(\varphi)$ has a right Engel sink $\mathcal{R}(g)$ of cardinality at most m . Then G has a nilpotent normal subgroup of index bounded in terms of m only.*

Recall that the Fitting series starts with the Fitting subgroup $F_1(G) = F(G)$, and by induction, $F_{k+1}(G)$ is the inverse image of $F(G/F_k(G))$. Throughout the paper we shall write, say, " (a, b, \dots) -bounded" to abbreviate "bounded above in terms of a, b, \dots only". Thus, the conclusion of Theorem 1.3 states that the index of $F_2(G)$

is m -bounded, and the conclusion of Theorem 1.4 states that the index of $F(G)$ is m -bounded.

The results of Theorems 1.3 and 1.4 are in a sense best-possible. There are easy examples showing that in Theorem 1.3 one cannot obtain a bound in terms of m (or even in terms of $|\varphi|$ and m) for the index of $F(G)$, nor can one prove that there is a normal subgroup of $(|\varphi|, m)$ -bounded order with metanilpotent quotient. In Theorem 1.3 one cannot prove that there is a normal subgroup of $(|\varphi|, m)$ -bounded order with nilpotent quotient.

We do not know whether the results of Theorems 1.3 and 1.4 can be extended to an automorphism of prime order not coprime to $|G|$. Another possible direction of further studies would be considering automorphisms of composite order.

In Sect. 3 we perform reduction of the proofs of both Theorems 1.3 and 1.4 to the case of soluble groups, using the classification of finite simple groups. Then in Sect. 4 the proof of Theorem 1.3 about left Engel sinks is completed: first a ‘weak’ upper bound for the index of $F_2(G)$ in terms of $|\varphi|$ and m is obtained, and then a ‘strong’ upper bound, in which the dependence on $|\varphi|$ is eliminated. The proof of Theorem 1.4 is completed in Sect. 5. Requisite preliminary material is collected in Sect. 2.

2 Preliminaries

For a group A acting by automorphisms on a group B we use the usual notation for commutators $[b, a] = b^{-1}b^a$ and commutator subgroups $[B, A] = \langle [b, a] \mid b \in B, a \in A \rangle$, as well as for centralizers $C_B(A) = \{b \in B \mid b^a = b \text{ for all } a \in A\}$. We usually denote by the same letter α the automorphism induced by an automorphism α on the quotient by a normal α -invariant subgroup.

We say for short that an automorphism φ of a finite group G is a coprime automorphism if the orders of φ and G are coprime: $(|G|, |\varphi|) = 1$. First we recall some properties of coprime automorphisms of finite groups. Several well-known facts about coprime automorphisms will be often used without special references, but we recall them in the following lemma.

Lemma 2.1 *Let φ be a coprime automorphism of a finite group G .*

- (a) *For every prime p dividing $|G|$, there is a φ -invariant Sylow p -subgroup of G .*
- (b) *If G is soluble, then for every set π of primes dividing $|G|$, there is a φ -invariant Hall π -subgroup of G .*
- (c) *If N is a normal φ -invariant subgroup of G , then the fixed points of φ in the quotient G/N are covered by the fixed points of φ in G , that is, $C_{G/N}(\varphi) = C_G(\varphi)N/N$.*

In the next useful lemma the condition that the group is nilpotent cannot be dropped.

Lemma 2.2 (Rodrigues and Shumyatsky 2020, Lemma 2.4) *Let φ be a coprime automorphism of a finite nilpotent group G . Then any element $g \in G$ can be uniquely written in the form $g = cu$, where $c \in C_G(\varphi)$ and $u = [v, \varphi]$ for some $v \in G$.*

As a special case, we have the following well-known fact.

Lemma 2.3 *Let φ be a coprime automorphism of an abelian finite group G . Then $G = [G, \varphi] \times C_G(\varphi)$.*

The following lemma appeared in Khukhro and Shumyatsky (2016) in a slightly less general form, so we reproduce the proof for the benefit of the reader.

Lemma 2.4 *Let V be an abelian finite group, and U a group of coprime automorphisms of V . If $[V, u] \leq m$ for every $u \in U$, then $[V, U]$ is m -bounded, and therefore $|U|$ is also m -bounded.*

Proof First suppose that U is abelian. Pick $u_1 \in U$ such that $[V, u_1] \neq 0$. By Lemma 2.3, $V = [V, u_1] \times C_V(u_1)$, and both summands are U -invariant, since U is abelian. If $C_U([V, u_1]) = 1$, then $|U|$ is m -bounded and $[V, U]$ has m -bounded order being generated by $[V, u]$, $u \in U$. Otherwise pick $1 \neq u_2 \in C_U([V, u_1])$; then $V = [V, u_1] \times [V, u_2] \times C_V(\langle u_1, u_2 \rangle)$. If $1 \neq u_3 \in C_U([V, u_1] \times [V, u_2])$, then $V = [V, u_1] \times [V, u_2] \times [V, u_3] \times C_V(\langle u_1, u_2, u_3 \rangle)$, and so on. If $C_U([V, u_1] \times \dots \times [V, u_k]) = 1$ at some m -bounded step k , then again $[V, U]$ has m -bounded order. However, if there are too many steps, then for the element $w = u_1 u_2 \dots u_k$ we shall have $0 \neq [V, u_i] = [[V, u_i], w]$, so that $[V, w] = [V, u_1] \times \dots \times [V, u_k]$ will have order greater than m , a contradiction.

We now consider the general case. Since every element $u \in U$ acts faithfully on $[V, u]$, the exponent of U is m -bounded. If P is a Sylow p -subgroup of U , let M be a maximal normal abelian subgroup of P . By the above, $[V, M]$ is m -bounded. Since M acts faithfully on $[V, M]$, we obtain that $|M|$ is m -bounded. Hence $|P|$ is m -bounded, since $C_P(M) = M$ and P/M embeds in the automorphism group of M . Since $|U|$ has only m -boundedly many prime divisors, it follows that $|U|$ is m -bounded. Since $[V, U] = \prod_{u \in U} [V, u]$, we obtain that $[V, U]$ is also m -bounded. □

Along with the aforementioned celebrated theorem of Thompson on the nilpotency of finite groups with a fixed-point-free automorphism of prime order, we shall also use the following theorem of Hartley and Meixner about an ‘almost fixed-point-free’ automorphism.

Theorem 2.5 (Hartley and Meixner 1981) *If a finite soluble group G admits an automorphism φ of prime order p such that $|C_G(\varphi)| = s$, then the index of $F(G)$ is (p, s) -bounded.*

We shall use the following lemma on actions of Frobenius groups.

Lemma 2.6 (Khukhro et al. 2014, Lemma 2.4) *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then $G = \langle C_G(H)^f \mid f \in F \rangle$.*

We denote by $\gamma_\infty(G) = \bigcap_i \gamma_i(G)$ the intersection of the lower central series of a group G ; when G is finite, then $G/\gamma_\infty(G)$ is the largest nilpotent quotient of G . We note the following elementary fact.

Lemma 2.7 *If $|\gamma_\infty(G)| = n$, then the centralizer $C_G(\gamma_\infty(G))$ is a nilpotent normal subgroup of n -bounded index, so that $|G/F(G)|$ is n -bounded.*

Proof Indeed, for any $c_i \in C_G(\gamma_\infty(G))$ a long enough commutator $[c_1, \dots, c_k]$ belongs to $\gamma_\infty(G)$ and then $[c_1, \dots, c_k, c_{k+1}] = 1$. Thus, $C_G(\gamma_\infty(G)) \leq F(G)$. The quotient $G/C_G(\gamma_\infty(G))$ embeds into the automorphism group of $\gamma_\infty(G)$ and therefore also has n -bounded order. Hence $|G/F(G)|$ is also n -bounded. \square

We now recall some elementary properties of Engel sinks. In cases where we need to consider the left or right Engel sink constructed with respect to a subgroup H containing g , we write $\mathcal{E}_H(g)$ or $\mathcal{R}_H(g)$, respectively. The following lemma is a consequence of the properties of coprime actions and a formula in Heineken's paper (Heineken 1960) on a connection between left and right Engel commutators.

Lemma 2.8 (Khukhro and Shumyatsky 2019, Lemma 3.2) *If V is an abelian subgroup of a finite group G , and $g \in G$ an element normalizing V such that $(|V|, |g|) = 1$, then*

$$[V, g] = \mathcal{E}_{V\langle g \rangle}(g) = \mathcal{R}_{V\langle g \rangle}(g).$$

Remark 2.9 In view of Lemma 2.1(c), the hypotheses of Theorems 1.3 and 1.4 are inherited by any φ -invariant section, that is, a quotient A/B of a φ -invariant subgroup A by a φ -invariant subgroup B normal in A . We shall freely use this fact without special references.

3 Bounding the Index of the Soluble Radical

In this section we perform a reduction of the proof of Theorems 1.3 and 1.4 to the case of soluble groups. For that we use a recent result of Guralnick and Tracey (2020).

Theorem 3.1 (Guralnick and Tracey (2020), Corollary 1.9) *Let $\alpha \in \text{Aut } G$ be an involutive automorphism of a finite group G , and let $J(\alpha) = \{g \in G \mid g \text{ has odd order and } g^\alpha = g^{-1}\}$. If $G = [G, \alpha]$, then the index of the Fitting subgroup $F(G)$ is bounded in terms of $|J(\alpha)|$, namely, $|G/F(G)| \leq |J(\alpha)|!^4$.*

In fact, we can derive from Theorem 3.1 a general proposition about groups of coprime automorphisms whose fixed points have bounded Engel sinks. We state this proposition here in greater generality than needed in the present paper, as it may find applications in other studies.

Proposition 3.2 *Let G be a finite group admitting a group of coprime automorphisms H . Suppose that m is a positive integer such that every 2-element in $C_G(H)$ has a left (or right) Engel sink of cardinality at most m . Then the soluble radical $S(G)$ of G has m -bounded index.*

(In the statement, “left (or right)” is applied individually to the 2-elements of $C_G(H)$, so that it may be different, left or right, Engel sinks for different 2-elements of $C_G(H)$.)

Proof Since the hypothesis is inherited by $G/S(G)$, we can assume that $S(G) = 1$. Then the generalized Fitting subgroup

$$F^*(G) = S_1 \times \cdots \times S_n \tag{3.1}$$

is a direct product of non-abelian finite simple groups S_i , which are permuted by H . Since the centralizer of $F^*(G)$ is trivial and G embeds into the automorphism group of $F^*(G)$, it is sufficient to obtain a bound for $|F^*(G)|$ in terms of m . We can simply assume that $G = F^*(G)$.

As proved by Wang and Chen (1993) on the basis of the classification of finite simple groups, a finite group admitting a group of coprime automorphisms with fixed-point subgroup of odd order is soluble. Therefore $C_G(H)$ contains involutions. Moreover, we can choose an involution $\alpha \in C_G(H)$ with non-trivial projections onto each of the factors S_i in (3.1). Indeed, let $\{S_{i_1}, \dots, S_{i_k}\}$ be one of the orbits of H in its permutational action on the set $\{S_1, \dots, S_n\}$. Clearly, any non-trivial element of the product $S_{i_1} \times \cdots \times S_{i_k}$ centralized by H must have non-trivial projections onto each of the factors S_{i_j} . There is an involution in the centralizer of H in the product $S_{i_1} \times \cdots \times S_{i_k}$. We now take $\alpha \in C_G(H)$ to be the product of these involutions over all orbits of H . Then $\alpha \in C_G(H)$ is an involution with non-trivial projections onto each of the factors S_i in (3.1). It follows that $G = F^*(G) = [G, \alpha]$.

If $g \in J(\alpha)$, that is, $g^\alpha = g^{-1}$, then $[\langle g \rangle, \alpha] = \langle g \rangle$, since g has odd order. Then $\langle g \rangle \subseteq \mathcal{R}(\alpha) \cap \mathcal{E}(\alpha)$ by Lemma 2.8. Therefore, $J(\alpha) \subseteq \mathcal{R}(\alpha) \cap \mathcal{E}(\alpha)$, so that $|J(\alpha)| \leq m$. We also have $[G, \alpha] = G$ and $F(G) = 1$, and Theorem 3.1 completes the proof. \square

In our situation, as an obvious consequence we obtain the following.

Corollary 3.3 *Let G be a finite group admitting a coprime automorphism φ of prime order. Suppose that m is a positive integer such that one of the following conditions holds (or both):*

- (a) *every element $g \in C_G(\varphi)$ has a left Engel sink $\mathcal{E}(g)$ of cardinality at most m , or*
- (b) *every element $g \in C_G(\varphi)$ has a right Engel sink $\mathcal{R}(g)$ of cardinality at most m .*

Then the soluble radical $S(G)$ has m -bounded index in G .

4 Completion of the Proof for Left Engel Sinks

Proof of Theorem 1.3 Recall that G is a finite group admitting an automorphism φ of prime order p coprime to $|G|$, and every element of $C_G(\varphi)$ has a left Engel sink of cardinality at most m ; we need to prove that $|G/F_2(G)|$ is m -bounded. By Corollary 3.3(a) the index of the soluble radical $|G/S(G)|$ is m -bounded. Therefore we can assume that G is soluble.

First we obtain a ‘weak’ upper bound for $|G/F_2(G)|$ in terms of p and m .

Lemma 4.1 *The index of $F_2(G)$ is (p, m) -bounded.*

Proof By Gaschütz’s theorem the image of the Fitting subgroup $F(G)$ in the quotient $G/\Phi(G)$ of G by the Frattini subgroup $\Phi(G)$ is the Fitting subgroup of this quotient (see Robinson 1996, 5.2.15), and hence the same holds for all terms of the Fitting series. Therefore, since the hypothesis is inherited by $G/\Phi(G)$, we can assume from the outset that $\Phi(G) = 1$. Then $F(G)$ is abelian by Gaschütz’s theorem.

To lighten the notation, let $F_i = F_i(G)$. Since F_1 is abelian, every element $g \in F_1C_{F_2}(\varphi)$ has Engel sink of cardinality at most m . Indeed, let $g = uc$, where $u \in F_1$ and $c \in C_{F_2}(\varphi)$. For any $h \in G$, a long enough commutator $d = [h, {}_k g]$ belongs to F_1 , since F_2/F_1 is nilpotent. Then $[d, {}_n g] = [d, {}_n uc] = [d, {}_n c]$ for any n , since F_1 is abelian. As a result, $\mathcal{E}(g)$ is contained in $\mathcal{E}(c)$, which has cardinality at most m by hypothesis.

Applying Theorem 1.1, we now obtain that $\gamma_\infty(F_1C_{F_2}(\varphi))$ has m -bounded order. It follows that the Fitting subgroup $F(F_1C_{F_2}(\varphi))$ has m -bounded index in $F_1C_{F_2}(\varphi)$ by Lemma 2.7. But the Fitting subgroup $F(F_1C_{F_2}(\varphi))$ is equal to F_1 , since $F_1C_{F_2}(\varphi)$ is a subnormal subgroup of G . Hence $C_{F_2/F_1}(\varphi) = C_{F_2}(\varphi)F_1/F_1$ has m -bounded order.

We can apply the same arguments to the quotient of G/F_1 by its Frattini subgroup. We obtain that $C_{F_3/F_2}(\varphi)$ also has m -bounded order. As a result, $C_{F_3/F_1}(\varphi)$ has m -bounded order. By Theorem 2.5 then the Fitting subgroup $F(F_3/F_1)$ has (p, m) -bounded index in F_3/F_1 . But, obviously, $F(F_3/F_1) = F_2/F_1$, so that F_3/F_2 has (p, m) -bounded order. Since G is soluble, F_3/F_2 contains its centralizer in G/F_2 . Then the group G/F_2 embeds into the automorphism group of F_3/F_2 and therefore also has (p, m) -bounded order. □

We proceed with the proof of Theorem 1.3, where we need to obtain a ‘strong’ bound, in terms of m only, for $|G/F_2(G)|$. If $p \leq m$, then the bound for $|G/F_2(G)|$ in terms of p and m obtained in Lemma 4.1 is a required bound in terms of m only. Therefore we can assume that $p > m$. We observe that when $x \in C_G(\varphi)$, the Engel sink $\mathcal{E}(x)$ is φ -invariant. Then the condition $p > m \geq |\mathcal{E}(x)|$ implies that

$$\mathcal{E}(x) \subseteq C_G(\varphi) \quad \text{for every } x \in C_G(\varphi). \tag{4.1}$$

Let $N = \langle C_G(\varphi)^G \rangle$ be the normal closure of $C_G(\varphi)$. Our goal now is to prove that $\gamma_\infty(N)$ has m -bounded order. The bulk of the proof is in the following key lemma.

Lemma 4.2 *We have $\gamma_\infty(N) \leq C_G(\varphi)$.*

Proof We use induction on the order of G . To lighten the notation, we write $\gamma = \gamma_\infty(N)$.

Recall that G is soluble. Let V be a minimal normal subgroup of $G\langle\varphi\rangle$, so it is an elementary abelian q -group for some prime q . By induction applied to G/V , we have $\gamma \leq VC_G(\varphi)$. By the minimality of V , either $\gamma \cap V = 1$ or $\gamma \geq V$. If $\gamma \cap V = 1$, then $[\gamma, \varphi] \leq [VC_G(\varphi), \varphi] = [V, \varphi] \cap \gamma = 1$, whence $\gamma \leq C_G(\varphi)$ and the proof is complete. Hence we can assume that $\gamma \geq V$.

We have $V = [V, \varphi] \times C_V(\varphi)$ by Lemma 2.3. The subgroup $[V, \varphi]$ is $C_G(\varphi)$ -invariant. Hence, $\gamma = [V, \varphi](C_G(\varphi) \cap \gamma)$. Any q' -element x of $C_G(\varphi)$ acts trivially on $[V, \varphi]$ by Lemma 2.8 because $\mathcal{E}(x) \subseteq C_G(\varphi)$ by (4.1). We record this property in the form

$$[\gamma, x] \leq C_G(\varphi) \quad \text{for any } q' \text{ - element } x \text{ of } C_G(\varphi). \tag{4.2}$$

Any Hall q' -subgroup of γ has the form H^v , where $v \in V$ and H is a Hall q' -subgroup of $C_G(\varphi) \cap \gamma$. Let $v = v_1v_2$ for $v_1 \in [V, \varphi]$ and $v_2 \in C_V(\varphi)$. Since $[H, v_1] = 1$ by (4.2), we have $H^v = H^{v_2} \leq C_G(\varphi) \cap \gamma$. As a result, the subgroup $O^q(\gamma)$ generated by all its Hall q' -subgroups is contained in $C_G(\varphi) \cap \gamma$. If $O^q(\gamma) \neq 1$, then by induction $\gamma \leq O^q(\gamma)C_G(\varphi) \leq C_G(\varphi)$ and the proof is complete. Therefore we can assume that $O^q(\gamma) = 1$, that is, γ is a q -group.

If $\Phi(\gamma) \neq 1$, then $\gamma \leq \Phi(\gamma)C_V(\varphi)$ by induction. But

$$\Phi(\gamma) = [[V, \varphi], C_G(\varphi) \cap \gamma] \cdot \Phi(C_G(\varphi) \cap \gamma),$$

so then $[V, \varphi] \leq [[V, \varphi], C_G(\varphi) \cap \gamma]$, where the right-hand side is strictly smaller than $[V, \varphi]$, since γ is nilpotent. This is a contradiction, unless $[V, \varphi] = 1$, when the proof is complete. Hence we can assume that $\Phi(\gamma) = 1$, so that γ is an elementary abelian q -group.

Let R be a φ -invariant Hall q' -subgroup of N . Then $[\gamma, R] = \gamma$ and therefore, by Lemma 2.3,

$$C_\gamma(R) = 1, \tag{4.3}$$

since γ is abelian.

We now consider γ as a right $\mathbb{F}_q G\langle\varphi\rangle$ -module and extend the ground field to a finite splitting field k of $G\langle\varphi\rangle$; let M be the resulting $kG\langle\varphi\rangle$ -module. The additive group of M is a finite q -group, and therefore both R and φ act by coprime automorphisms, to which Lemma 2.1(c) on covering fixed points applies. Note that by (4.3) we have

$$C_M(R) = 0. \tag{4.4}$$

We continue using the same notation for centralizers and commutator subgroups, albeit using the additive structure of M .

We now consider an unrefinable series of $kG\langle\varphi\rangle$ -submodules connecting 0 with M . We consider an arbitrary factor U of this series, which is an irreducible $kG\langle\varphi\rangle$ -module. Let the bar denote the images of elements and subgroups in the action on U . Note that $\bar{\gamma} = 1$, so that \bar{N} is nilpotent. We can sometimes drop the bar when considering the action of $G\langle\varphi\rangle$ on U ; for example,

$$C_U(R) = 0 \tag{4.5}$$

by (4.4) and Lemma 2.1. We also have

$$[U, C_R(\varphi)] \leq C_U(\varphi) \tag{4.6}$$

by (4.2).

Let $U = W_1 \oplus \dots \oplus W_n$ be the decomposition of U into the sum of Wedderburn homogeneous components with respect to \bar{R} , which is a normal subgroup of \bar{G} , since \bar{N} is nilpotent. By Clifford's theorem, the group $G\langle\varphi\rangle$ transitively permutes the components W_i , and the kernel of this permutational action contains $RC_G(R) \geq N$.

We claim that $C_R(\varphi)$ acts non-trivially on one of the components W_i . Indeed, otherwise $C_R(\varphi)$ acts trivially on U . But \bar{R} is the normal closure of $C_{\bar{R}}(\varphi)$ because \bar{N}

is nilpotent and is the normal closure of $C_{\bar{G}}(\varphi)$. Thus we obtain that R acts trivially on U , which contradicts (4.5).

For definiteness, let W_1 be a component on which $C_R(\varphi)$ acts non-trivially. By (4.6) then $[W_1, C_R(\varphi)]$ (which is contained in W_1) is a non-trivial subspace of $C_U(\varphi)$. Hence φ belongs to the stabilizer of W_1 . Since $\bar{N} \leq \bar{R}C_{\bar{G}}(\bar{R})$ is in the kernel of the permutational action on the set of the W_i and φ acts fixed-point-freely on G/N , all orbits of φ on the set of the W_i have length p except the one-element orbit $\{W_1\}$. Indeed, let S be the stabilizer of W_1 . If $g \notin S$ and $W_1g\varphi = W_1g$, then, since $\varphi \in S$, the coset Sg is φ -invariant. Since $|Sg|$ is coprime to $p = |\varphi|$, then φ would have a fixed point in Sg , contrary to $C_G(\varphi) \leq N \leq S$.

We claim that φ actually cannot have any orbits of length p on the set of the W_i , so that $\{W_1\}$ is the only orbit of φ , which of course means that U is a homogeneous $k\bar{R}$ -module. Suppose the opposite, and let $W_1t, W_1t^\varphi, \dots, W_1t^{\varphi^{p-1}}$ be a regular orbit of φ (here, $W_1t^{\varphi^i} = W_1t\varphi^i$, since $\varphi \in S$). Then $C_R(\varphi)$ must centralize all these $W_1t^{\varphi^i}$ because of (4.6):

$$[W_1t^{\varphi^i}, C_R(\varphi)] = 0 \quad \text{for } i = 0, 1, \dots, p - 1. \tag{4.7}$$

For every $i = 0, 1, \dots, p - 1$, the subgroup $C_R(\varphi)^{t^{\varphi^i}}$ acts nontrivially on $W_1t^{\varphi^i}$ and centralizes all the other $W_1t^{\varphi^s}$ for $s \neq i$. Let tilde denote the images in the action on the direct sum

$$W_1t \oplus W_1t^\varphi \oplus \dots \oplus W_1t^{\varphi^{p-1}}.$$

Then the subgroups $\widetilde{C_R(\varphi)^t}, \widetilde{C_R(\varphi)^{t^\varphi}}, \dots, \widetilde{C_R(\varphi)^{t^{\varphi^{p-1}}}}$ commute and generate a direct product

$$\widetilde{C_R(\varphi)^t} \times \widetilde{C_R(\varphi)^{t^\varphi}} \times \dots \times \widetilde{C_R(\varphi)^{t^{\varphi^{p-1}}}},$$

in which the factors are permuted by φ . The diagonal of this direct product is an image of a subgroup of $C_R(\varphi)$ and acts non-trivially on these $W_1^t, W_1^{t^\varphi}, \dots, W_1^{t^{\varphi^{p-1}}}$, contrary to (4.7). This contradiction shows that φ has no orbits of length p .

Thus, $\{W_1\}$ is the only orbit of φ , so that $U = W_1$ is a homogeneous $k\bar{R}$ -module. Since \bar{R} is nilpotent and k is a splitting field, the centre $Z(\bar{R})$ acts by scalar multiplications on W_1 . Hence $Z(\bar{R})$ commutes with $\bar{\varphi}$. By Lemma 2.1(c) there is an element $z \in C_R(\varphi)$ whose image is a non-trivial element of $Z(\bar{R})$. Since $[U, z] \leq C_U(\varphi)$ by (4.6) and $U = [U, z]$, we obtain that $[U, \varphi] = 0$.

Since the above arguments apply to every irreducible factor of that series in M , we obtain by Lemma 2.1(c) that $[M, \varphi] = 0$, which means that $\gamma \leq C_G(\varphi)$, as required. □

Lemma 4.3 *The order of $\gamma_\infty(N)$ is m -bounded.*

Proof By Lemma 2.2 every element of $N/\gamma_\infty(N)$ can be written as a product of a commutator $[h, \varphi]$ for $h \in N/\gamma_\infty(N)$ and an element from $C_{N/\gamma_\infty(N)}(\varphi) =$

$C_N(\varphi)\gamma_\infty(N)/\gamma_\infty(N)$. Since $\gamma_\infty(N) \leq C_N(\varphi)$ by Lemma 4.2, it follows that every element $n \in N$ can be written as a product $n = [g, \varphi]c$ for $g \in N$ and $c \in C_N(\varphi)$.

We now claim that $\mathcal{E}(n) = \mathcal{E}(c)$. This follows from the fact that $[g, \varphi]$ centralizes $\gamma_\infty(N)$. Indeed, since $\gamma_\infty(N) \leq C_N(\varphi)$ by Lemma 4.2 and $\gamma_\infty(N)$ is a normal subgroup of $G\langle\varphi\rangle$, the centralizer $C_G(\gamma_\infty(N))$ is a normal subgroup that contains φ and therefore also contains $[g, \varphi]$.

The quotient G/N admits a fixed-point-free automorphism φ of prime order and therefore is nilpotent by Thompson’s theorem (Thompson 1959). Therefore for any $x \in G$ we have $[x, {}_s n] \in N$ for some s , and then $[x, {}_{s+t} n] \in \gamma_\infty(N)$ for some t . Since $[g, \varphi]$ centralizes $\gamma_\infty(N)$ as shown above, we further have $[x, {}_{s+t+u} n] = [[x, {}_{s+t} n], {}_u c]$ and this element belongs to $\mathcal{E}(c)$ for large enough u .

As a result, all elements of N have Engel sinks of size at most m and therefore $\gamma_\infty(N)$ has m -bounded order by Theorem 1.1. □

We now finish the proof of Theorem 1.3. In the remaining case $p > m$ we have $\gamma_\infty(N)$ of m -bounded order by Lemma 4.3. Then $C_G(\gamma_\infty(N))$ has m -bounded index and is metanilpotent, since $C_G(\gamma_\infty(N))/(N \cap C_G(\gamma_\infty(N)))$ is nilpotent and $N \cap C_G(\gamma_\infty(N))$ is nilpotent by Lemma 2.7. □

5 Completion of the Proof for Right Engel Sinks

Proof of Theorem 1.4 Recall that G is a finite group admitting an automorphism φ of prime order p coprime to $|G|$, and every element of $C_G(\varphi)$ has a right Engel sink of cardinality at most m ; we need to prove that $|G/F(G)|$ is m -bounded. By Corollary 3.3(b) the index of the soluble radical $|G/S(G)|$ is m -bounded. Therefore we can assume that G is soluble.

Since $G/F_2(G)$ embeds in the automorphism group of $F_2(G)/F(G)$, we can assume that $G = F_2(G)$, so that G/F is nilpotent. We choose Thompson’s critical subgroup C_p in each Sylow p -subgroup P of G/F , which is a characteristic subgroup of P such that $C_p/Z(C_p)$ is an elementary abelian p -group and $C_P(C_p) = Z(C_p)$ (see Gorenstein 1980, Theorem 5.3.11). Then the product $C = \prod_p C_p$ is a characteristic subgroup of G/F such that $C/Z(C)$ is a direct product of elementary abelian groups and $C_{G/F}(C) = Z(C)$. Since $(G/F)/Z(C)$ embeds into the automorphism group of C , it is sufficient to prove that $|C|$ is m -bounded. Therefore we can replace G by the inverse image of C and assume that $G/F = C$.

Since $F(G)/Z(G) = F(G/Z(G))$ we can assume that $Z(G) = 1$.

Consider the Fitting subgroup $F(G\langle\varphi\rangle)$ of the semidirect product $G\langle\varphi\rangle$. We have $F(G\langle\varphi\rangle) \cap G = F(G)$, so it suffices to show that the index of $F(G\langle\varphi\rangle)$ in $G\langle\varphi\rangle$ is m -bounded. By Gaschütz’s theorem (Huppert 1967, Satz III.4.2) the image of the Fitting subgroup in the quotient of $F(G\langle\varphi\rangle)$ by its Frattini subgroup is the Fitting subgroup of this quotient. Since the hypothesis is inherited by this quotient, we can assume that the Frattini subgroup of $F(G\langle\varphi\rangle)$ is trivial. Then $F(G\langle\varphi\rangle)$ is a direct product of minimal normal subgroups of $G\langle\varphi\rangle$ by Gaschütz’s theorem (Huppert 1967, Satz III.4.5). Therefore $F(G)$ is a direct product of minimal normal φ -invariant subgroups, which

are elementary abelian q -groups for various q . To lighten the notation, in what follows we write $F = F(G)$.

Lemma 5.1 *The centralizer $C_{G/F}(\varphi)$ of φ in G/F has m -bounded order.*

Proof Let q be any prime dividing $|C_{G/F}(\varphi)|$, and let Q be a Sylow q -subgroup of $C_{G/F}(\varphi)$. Consider any non-trivial element $g \in Q$, and let \hat{g} be a q -element of G that is a pre-image of g chosen in $C_G(\varphi)$ in accordance with Lemma 2.2. Then \hat{g} induces by conjugation a non-trivial coprime automorphism of the Hall q' subgroup $F_{q'}$ of F . Indeed, if \hat{g} centralized $F_{q'}$, then \hat{g} would centralize all factors of a principal series of G and would belong to F by Robinson (1996, 5.2.9). We have $[F_{q'}, \hat{g}] \subseteq \mathcal{R}(\hat{g})$ by Lemma 2.8. Hence $[F_{q'}, \hat{g}] = [F_{q'}, g]$ has m -bounded order for any $g \in Q$. Then $|Q|$ is m -bounded by Lemma 2.4. In particular, the prime q is bounded above in terms of m . As a result, $|C_{G/F}(\varphi)|$ is m -bounded. \square

Due to Lemma 5.1 we can use induction on $|C_{G/F}(\varphi)|$ in the proof of Theorem 1.4 to show that we can assume that $C_{G/F}(\varphi) = 1$. Namely, if $C_{G/F}(\varphi) \neq 1$, then for some prime q and a Sylow q -subgroup Q of G/F , there is a non-trivial element $c \in C_Q(\varphi)$. As before, $[F_{q'}, c] \neq 1$. If $c \notin Z(Q)$, then, since $Q/Z(Q)$ is elementary abelian, by Maschke’s theorem there is a normal φ -invariant subgroup Q_1 of Q of index q not containing c . Since here q is m -bounded, the inverse image G_1 of the product of Q_1 and the Hall q' -subgroup of G/F is a φ -invariant subgroup of m -bounded index with $F(G_1) = F$ such that $|C_{G_1/F}(\varphi)| < |C_{G/F}(\varphi)|$. Then the induction hypothesis applied to G_1 completes the proof. If, however, $c \in Z(Q)$, then let \hat{c} be a q -element of G that is a pre-image of c chosen in $C_G(\varphi)$ in accordance with Lemma 2.2. Then $[F_{q'}, c]$ is a normal φ -invariant subgroup of G , since $c \in Z(G/F)$ and F is nilpotent. Furthermore, $[F_{q'}, c] \subseteq \mathcal{R}(\hat{c})$ by Lemma 2.8, and therefore $[F_{q'}, c]$ has m -bounded order. We obtain that $C_G([F_{q'}, c])$ is a φ -invariant subgroup of m -bounded index, which does not contain c , since $[[F_{q'}, c], c] = [F_{q'}, c] \neq 1$. Then the induction hypothesis applied to $C_G([F_{q'}, c])$ completes the proof.

Thus, we can assume that $C_{G/F}(\varphi) = 1$ for the rest of the proof. This means that $(G/F)\langle\varphi\rangle$ is a Frobenius group with kernel G/F and complement $\langle\varphi\rangle$ acting on F . It follows by Lemma 2.6 that every φ -invariant normal section S of G such that $C_S(G/F) = 1$ is the normal closure of $C_S(\varphi)$ under the action of G :

$$S = \langle C_S(\varphi)^g \mid g \in G \rangle \quad \text{if } C_S(G/F) = 1. \tag{5.1}$$

Recall that F is a direct product of minimal normal φ -invariant subgroups (which are elementary abelian q -groups for various, not necessarily different, primes q):

$$F = V_1 \times \cdots \times V_k. \tag{5.2}$$

Let V be one of these factors V_i , and let $\tilde{G} = G/C_G(V)$ be the image of G in the action by conjugation on V . We record some properties of the action of G on V in the following lemma.

Lemma 5.2 *If V is an elementary abelian q -group, then \tilde{G} is a non-trivial q' -group and $C_V(G) = 1$.*

Proof Indeed, suppose that Q is a nontrivial Sylow q -subgroup of \bar{G} . Since \bar{G} is nilpotent, then $Z(VQ) \cap V$ is a proper G -invariant subgroup of V , which is also φ -invariant. This contradicts the minimality of V . Since $Z(G) = 1$ by our assumption, we have $\bar{G} \neq 1$. Since $C_V(G)$ is normal in $G\langle\varphi\rangle$, we must have $C_V(G) = 1$ by the minimality of V . \square

We now prove a key lemma in the proof of Theorem 1.4.

Lemma 5.3 *For any factor $V = V_i$ in the product (5.2), the order of $\bar{G} = G/C_G(V)$ is m -bounded.*

Proof Let V be an elementary abelian q -group. Since $\bar{G} \neq 1$, we have $V = \langle C_V(\varphi)^g \mid g \in G \rangle$ by (5.1). Let $1 \neq c \in C_V(\varphi)$. For any $x \in \bar{G}$ and any positive integer n we have $[c, q^n x] = [c, x^{q^n}]$. Indeed, regarding V as an $\mathbb{F}_q \bar{G}$ -module, we see that

$$[c, q^n x] = c(x - 1)^{q^n} = c(x^{q^n} - 1) = [c, x^{q^n}],$$

in characteristic q . For all large enough n the element $[c, q^n x]$ belongs to the right Engel sink $\mathcal{R}(c)$. Choosing n to be the same large enough integer for all elements $x \in \bar{G}$, we obtain that $[c, x^{q^n}]$ takes at most m values when x runs over \bar{G} , since $|\mathcal{R}(c)| \leq m$. Since \bar{G} is a finite q' -group by Lemma 5.2, the elements x^{q^n} run over the entire \bar{G} as x does. Thus, $[c, x]$ takes at most m values when x runs over \bar{G} , which means that the index of the centralizer $C_{\bar{G}}(c)$ in \bar{G} is at most m . The intersection $Z = \bigcap_{g \in \bar{G}} C_{\bar{G}}(c)^g$ is a G -invariant subgroup of m -bounded index, and this intersection is also φ -invariant, since $C_{\bar{G}}(c)$ is. Then $[c^g, Z] = 0$ for all $g \in G$. But $V = \langle c^g \mid g \in \bar{G} \rangle$, because V is a minimal normal subgroup of $G\langle\varphi\rangle$, while the right-hand side is both G - and φ -invariant. Hence, $[V, Z] = 0$ or, in other words, $Z = 1$, so that \bar{G} has m -bounded order. \square

We now finish the proof of Theorem 1.4. Recall that any element of G/F acts nontrivially on at least one of the factors V_i in (5.2). Let $1 \neq c_1 \in C_{V_1}(\varphi)$, which exists by (5.1). Then the right Engel sink of c_1 is non-trivial: $\mathcal{R}(c_1) \neq \{1\}$. Indeed, otherwise c_1 belongs to the hypercentre $\zeta_\infty(G)$ by Baer’s theorem (Robinson 1996, 12.3.7), and then $V_1 \leq \zeta_\infty(G)$ by the minimality of V_1 . This, however, contradicts Lemma 5.2. Therefore we can choose $a_1 \in G$ such that

$$[c_1, {}_n a_1] \neq 1 \quad \text{for any } n \in \mathbb{N}.$$

If $C_G(V_1) = F$, then $|G/F|$ is m -bounded by Lemma 5.3 and the proof is complete. Otherwise $C_G(V_1)$ acts non-trivially on at least one of the remaining factors in (5.2). Without loss of generality, we can assume that $[V_2, C_G(V_1)] \neq 1$. Let $1 \neq c_2 \in C_{V_2}(\varphi)$, which exists by (5.1). Then the right Engel sink of c_2 in the product $V_2 C_G(V_1)$ is non-trivial: $\mathcal{R}_{V_2 C_G(V_1)}(c_2) \neq \{1\}$. Otherwise c_2 belongs to the hypercentre $\zeta_\infty(V_2 C_G(V_1))$ of the semidirect product $V_2 C_G(V_1)$ by Baer’s theorem (Robinson 1996, 12.3.7), and since $V_2 C_G(V_1)$ is a normal subgroup of $G\langle\varphi\rangle$, then

$V_2 \leq \zeta_\infty(V_2 C_G(V_1))$ by the minimality of V_2 , which contradicts Lemma 5.2. Therefore we can choose $a_2 \in C_G(V_1)$ such that

$$[c_2, {}_n a_2] \neq 1 \quad \text{for any } n \in \mathbb{N}.$$

Note that at the same time $[c_1, a_2] = 1$.

If $C_G(V_1 V_2) = C_G(V_1) \cap C_G(V_2) = F$, then $|G/F|$ is m -bounded by Lemma 5.3 and the proof is complete. Otherwise $C_G(V_1 V_2)$ acts non-trivially on at least one of the remaining factors in (5.2), say on V_3 . Let $1 \neq c_3 \in C_{V_3}(\varphi)$, which exists by (5.1). The right Engel sink of c_3 in the product $V_3 C_G(V_1 V_2)$ is non-trivial: $\mathcal{R}_{V_3 C_G(V_1 V_2)}(c_3) \neq \{1\}$, as otherwise $c_3 \in \zeta_\infty(V_3 C_G(V_1 V_2))$, whence $V_3 \leq \zeta_\infty(V_3 C_G(V_1 V_2))$ by the minimality of V_3 , contrary to Lemma 5.2. Therefore we can choose $a_3 \in C_G(V_1 V_2)$ such that

$$[c_3, {}_n a_3] \neq 1 \quad \text{for any } n \in \mathbb{N}.$$

Note that at the same time $[c_1, a_3] = 1$ and $[c_2, a_3] = 1$.

We can continue this construction in the obvious fashion as long as $C_G(V_1 V_2 \cdots V_k) \neq F$. We claim that this process will stop at $C_G(V_1 V_2 \cdots V_k) = F$ for some $k \leq m$. Indeed, otherwise we would have constructed elements $c_i \in V_i$ and $a_1 \in G$ and $a_i \in C_G(V_1 \cdots V_i)$ for $i = 2, \dots, m+1$ such that

$$[c_i, {}_n a_i] \neq 1 \quad \text{for any } n \in \mathbb{N}, \quad \text{while } [c_i, a_j] = 1 \quad \text{for any } j > i.$$

Then the product $c = c_1 c_2 \cdots c_{m+1}$ would have right Engel sink $\mathcal{R}(c)$ of cardinality at least $m+1$, since for any given $n_i \in \mathbb{N}$, $i = 1, 2, \dots, m+1$, the $m+1$ elements $[c, {}_{n_i} a_i]$ are different. Indeed, as an element of the product $V = V_1 \times \cdots \times V_k$, the commutator $[c, {}_{n_i} a_i]$ has trivial components in the factors V_1, \dots, V_{i-1} and a non-trivial component in V_i . The inequality $|\mathcal{R}(c)| \geq m+1$ contradicts the hypothesis of the theorem.

This contradiction shows that $C_G(V_1 V_2 \cdots V_k) = F$ for some $k \leq m$. Then

$$\begin{aligned} |G/F| &= |G/C_G(V_1 \cdots V_k)| = |G/(C_G(V_1) \cap \cdots \cap C_G(V_k))| \\ &\leq |G/C_G(V_1)| \cdots |G/C_G(V_k)| \end{aligned}$$

and we obtain that $|G/F|$ is m -bounded by Lemma 5.3. \square

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References

- Acciari, C., Silveira, D.: Profinite groups and centralizers of coprime automorphisms whose elements are Engel. *J. Group Theory* **21**(3), 485–509 (2018)
- Acciari, C., Silveira, D.: Engel-like conditions in fixed points of automorphisms of profinite groups. *Ann. Mat. Pura Appl.* (4) **199**(1), 187–197 (2020)
- Acciari, C., Shumyatsky, P., Silveira, D.S.: On groups with automorphisms whose fixed points are Engel. *Ann. Mat. Pura Appl.* **197**, 307–316 (2018)
- Acciari, C., Khukhro, E.I., Shumyatsky, P.: Profinite groups with an automorphism whose fixed points are right Engel. *Proc. Am. Math. Soc.* **147**(9), 3691–3703 (2019a)
- Acciari, C., Shumyatsky, P., Silveira, D.: Engel sinks of fixed points in finite groups. *J. Pure Appl. Algebra* **223**(11), 4592–4601 (2019b)
- Golod, E.S.: On nil-algebras and residually finite p -groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **28**, 273–276 (1964) [English transl. in *Amer. Math. Soc. Transl.* (2) **48**, 103–106 (1965)]
- Gorenstein, D.: *Finite Groups*, 2nd edn. Chelsea, London (1980)
- Guralnick, R.M., Tracey, G.: On the generalized Fitting height and insoluble length of finite groups. *Bull. Lond. Math. Soc.* **52**(5), 924–931 (2020)
- Hartley, B., Meixner, T.: Finite soluble groups containing an element of prime order whose centralizer is small. *Arch. Math. (Basel)* **36**, 211–213 (1981)
- Heineken, H.: Eine Bemerkung über engelsche Elemente. *Arch. Math. (Basel)* **11**, 321 (1960)
- Huppert, B.: *Endliche Gruppen*, I. Springer, Berlin (1967)
- Khukhro, E.I., Shumyatsky, P.: Almost Engel finite and profinite groups. *Int. J. Algebra Comput.* **26**(5), 973–983 (2016)
- Khukhro, E.I., Shumyatsky, P.: Almost Engel compact groups. *J. Algebra* **500**, 439–456 (2018a)
- Khukhro, E.I., Shumyatsky, P.: Finite groups with Engel sinks of bounded rank. *Glasgow Math. J.* **60**(3), 695–701 (2018b)
- Khukhro, E.I., Shumyatsky, P.: Compact groups all elements of which are almost right Engel. *Q. J. Math.* **70**, 879–893 (2019)
- Khukhro, E.I., Shumyatsky, P.: Compact groups with countable Engel sinks. *Bull. Math. Sci.* **9**(2), 2050015 (2020a)
- Khukhro, E.I., Shumyatsky, P.: Compact groups in which all elements have countable right Engel sinks. *Proc. R. Soc. Edinb. Sect. A Math.* (2020b). <https://doi.org/10.1017/prm.2020.81>
- Khukhro, E.I., Shumyatsky, P.: On profinite groups with automorphisms whose fixed points have countable Engel sinks, submitted (2020c). [arXiv:2006.05959](https://arxiv.org/abs/2006.05959)
- Khukhro, E.I., Makarenko, N.Yu., Shumyatsky, P.: Frobenius groups of automorphisms and their fixed points. *Forum Math.* **26**, 73–112 (2014)
- Khukhro, E.I., Shumyatsky, P., Traustason, G.: Right Engel-type subgroups and length parameters of finite groups. *J. Austral. Math. Soc.* (2019). <https://doi.org/10.1017/S1446788719000181>
- Robinson, D.J.S.: *A Course in the Theory of Groups*. Springer, New York (1996)
- Rodrigues, S., Shumyatsky, P.: Exponent of a finite group admitting a coprime automorphism. *J. Pure Appl. Algebra* **224**(9), article 106370 (2020)
- Thompson, J.: Finite groups with fixed-point-free automorphisms of prime order. *Proc. Natl. Acad. Sci. U.S.A.* **45**, 578–581 (1959)
- Wang, Y.M., Chen, Z.M.: Solubility of finite groups admitting a coprime order operator group. *Boll. Un. Mat. Ital. A* (7) **7**(3), 325–331 (1993)
- Wilson, J.S., Zelmanov, E.I.: Identities for Lie algebras of pro- p groups. *J. Pure Appl. Algebra* **81**(1), 103–109 (1992)