A PRO-p GROUP WITH INFINITE NORMAL HAUSDORFF SPECTRA

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Abstract. Using wreath products, we construct a finitely generated pro-p group G with infinite normal Hausdorff spectrum

$$\mathrm{hspec}^{\mathcal{P}}_{\trianglelefteq}(G) = \{\mathrm{hdim}^{\mathcal{P}}_G(H) \mid H \trianglelefteq_{\mathrm{c}} G\};$$

here $\mathrm{hdim}^{\mathcal{P}}_G \colon \{X \mid X \subseteq G\} \to [0,1]$ denotes the Hausdorff dimension function associated to the p-power series $\mathcal{P}\colon G^{p^i}$, $i \in \mathbb{N}_0$. More precisely, we show that $\mathrm{hspec}^{\mathcal{P}}_{\trianglelefteq}(G) = [0, 1/3] \cup \{1\}$ contains an infinite interval; this settles a question of Shalev. Furthermore, we prove that the normal Hausdorff spectra $\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(G)$ with respect to other filtration series $\mathcal{S}$ have a similar shape. In particular, our analysis applies to standard filtration series such as the Frattini series, the lower p-series and the modular dimension subgroup series.

Lastly, we pin down the ordinary Hausdorff spectra $\mathrm{hspec}^{\mathcal{S}}(G) = \{\mathrm{hdim}^{\mathcal{S}}_G(H) \mid H \leq_{\mathrm{c}} G\}$ with respect to the standard filtration series $\mathcal{S}$. The spectrum $\mathrm{hspec}^{\mathcal{L}}(G)$ for the lower p-series $\mathcal{L}$ displays surprising new features.

1. Introduction

The concept of Hausdorff dimension has led to interesting applications in the context of profinite groups; see [4] and the references given therein. Let G be a countably based infinite profinite group and consider a *filtration series* $\mathcal{S}$ of G, that is, a descending chain $G = G_0 \supseteq G_1 \supseteq \ldots$ of open normal subgroups $G_i \trianglelefteq_{\mathrm{o}} G$ such that $\bigcap_i G_i = 1$. These open normal subgroups form a base of neighbourhoods of the identity and induce a translation-invariant metric on G given by $d^{\mathcal{S}}(x,y) = \inf\{|G : G_i|^{-1} \mid x \equiv y \pmod{G_i}\}$, for $x, y \in G$. This, in turn, supplies the *Hausdorff dimension* $\mathrm{hdim}^{\mathcal{S}}_G(U) \in [0,1]$ of any subset $U \subseteq G$, with respect to the filtration series $\mathcal{S}$.

Barnea and Shalev [1] established the following 'group-theoretic' interpretation of the Hausdorff dimension of a closed subgroup H of G as a logarithmic density:

$$\mathrm{hdim}^{\mathcal{S}}_G(H) = \varliminf_{i \to \infty} \frac{\log|HG_i : G_i|}{\log|G : G_i|}.$$

The *Hausdorff spectrum* of G, with respect to $\mathcal{S}$, is

$$\mathrm{hspec}^{\mathcal{S}}(G) = \{\mathrm{hdim}^{\mathcal{S}}_G(H) \mid H \leq_{\mathrm{c}} G\} \subseteq [0,1],$$

where H runs through all closed subgroups of G. As indicated by Shalev in [7, §4.7], it is also natural to consider the *normal Hausdorff spectrum* of G, with

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respect to $\mathcal{S}$, namely

$$\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(G) = \{\mathrm{hdim}^{\mathcal{S}}_G(H) \mid H \trianglelefteq_\mathrm{c} G\}$$

which reflects the range of Hausdorff dimensions of closed normal subgroups. Apart from the observations in [7, §4.7], very little appears to be known about normal Hausdorff spectra of profinite groups.

Throughout we will be concerned with pro-p groups, where p denotes an odd prime; in Appendix A we indicate how our results extend to $p = 2$. We recall that even for well structured groups, such as p-adic analytic pro-p groups G, the Hausdorff dimension function and the Hausdorff spectrum of G are known to be sensitive to the choice of $\mathcal{S}$; compare [4]. However, for a finitely generated pro-p group G there are natural choices for $\mathcal{S}$, such as the p-power series $\mathcal{P}$, the Frattini series $\mathcal{F}$, the lower p-series $\mathcal{L}$ and the modular dimension subgroup series $\mathcal{D}$; see Section 2.

In this paper, we are interested in a particular group G constructed as follows. The pro-p wreath product $W = C_p \mathbin{\hat{\wr}} \mathbb{Z}_p$ is the inverse limit $\varprojlim_{k\in\mathbb{N}} C_p \wr C_{p^k}$ of the finite standard wreath products of cyclic groups with respect to the natural projections; clearly, W is 2-generated as a topological group. Let F be the free pro-p group on two generators and let $R \trianglelefteq_\mathrm{c} F$ be the kernel of a presentation $\pi \colon F \to W$. We are interested in the pro-p group

$$G = F/N, \qquad \text{where} \quad N = [R, F]R^p \trianglelefteq_\mathrm{c} F.$$

Up to isomorphism, the group G does not depend on the particular choice of π, as can be verified using Gaschütz' Lemma; see [6, Prop. 2.2]. Indeed, G can be described as the universal covering group for 2-generated central extensions of elementary abelian pro-p groups by W, i.e., for 2-generated pro-p groups E admitting a central elementary abelian subgroup A such that $E/A \cong W$.

Theorem 1.1. *For $p > 2$, the normal Hausdorff spectra of the pro-p group G constructed above, with respect to the standard filtration series $\mathcal{P}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{L}$ respectively, satisfy:*

$$\begin{aligned}
&\mathrm{hspec}^{\mathcal{P}}_{\trianglelefteq}(G) = \mathrm{hspec}^{\mathcal{D}}_{\trianglelefteq}(G) = [0, 1/3] \cup \{1\},\\
&\mathrm{hspec}^{\mathcal{F}}_{\trianglelefteq}(G) = [0, 1/(1+p)] \cup \{1\},\\
&\mathrm{hspec}^{\mathcal{L}}_{\trianglelefteq}(G) = [0, 1/5] \cup \{3/5\} \cup \{1\}.
\end{aligned}$$

In particular, they each contain an infinite real interval.

This solves a problem posed by Shalev [7, Problem 16]. We observe that the normal Hausdorff spectrum of G is sensitive to changes in filtration and that the normal Hausdorff spectrum of G with respect to the Frattini series varies with p.

In Section 4 we show that finite direct powers $G \times \ldots \times G$ of the group G provide examples of normal Hausdorff spectra consisting of multiple intervals. Furthermore, the sequence $G \times \overset{m}{\cdots} \times G$, $m \in \mathbb{N}$, has normal Hausdorff spectra 'converging' to $[0, 1]$; compare Corollary 4.5. We highlight three natural problems.

Problem 1.2. Does there exist a finitely generated pro-p group H

(a) with countably infinite normal Hausdorff spectrum $\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(H)$,
(b) with full normal Hausdorff spectrum $\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(H) = [0, 1]$,
(c) such that 1 is not an isolated point in $\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(H)$,

for one or several of the standard series $\mathcal{S} \in \{\mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L}\}$?

We also compute the entire Hausdorff spectra of G with respect to the four standard filtration series, answering en route a question raised in [3, VIII.7.2].

Theorem 1.3. *For $p > 2$, the Hausdorff spectra of the pro-p group G constructed above, with respect to the standard filtration series, satisfy:*

$$\mathrm{hspec}^{\mathcal{P}}(G) = \mathrm{hspec}^{\mathcal{D}}(G) = \mathrm{hspec}^{\mathcal{F}}(G) = [0,1],$$
$$\mathrm{hspec}^{\mathcal{L}}(G) = [0, 4/5) \cup \{3/5 + 2m/5p^n \mid m, n \in \mathbb{N}_0 \text{ with } p^n/2 < m \leq p^n\}.$$

The qualitative shape of the spectrum $\mathrm{hspec}^{\mathcal{L}}(G)$, i.e., its decomposition into a continuous and a non-continuous, but dense part, is unprecedented and of considerable interest; in Corollary 2.11 we show that already the wreath product $W = C_p \,\hat{\wr}\, \mathbb{Z}_p$ has a similar Hausdorff spectrum with respect to the lower p-series.

Organisation. Section 2 contains preliminary results. In Section 3 we give an explicit presentation of the pro-p group G and describe a series of finite quotients G_k, $k \in \mathbb{N}$, such that $G = \varprojlim G_k$. In Section 4 we provide a general description of the normal Hausdorff spectrum of G and, with respect to certain induced filtration series, we generalise this to finite direct powers of G. In Section 5 we compute the normal Hausdorff spectrum of G with respect to the p-power series $\mathcal{P}$, and in Section 6 we compute the normal Hausdorff spectra of G with respect to the other three standard filtration series $\mathcal{D}, \mathcal{F}, \mathcal{L}$. In Section 7 we compute the entire Hausdorff spectra of G. Finally, in Appendix A we indicate how our results extend to the case $p = 2$.

Notation. Throughout, p denotes an *odd* prime, although some results hold also for $p = 2$, possibly with minor modifications; only in Appendix A we discuss the analogous pro-2 groups. We denote by $\varliminf_{i\to\infty} a_i$ the lower limit (limes inferior) of a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathbb{R} \cup \{\pm\infty\}$. Tacitly, subgroups of profinite groups are generally understood to be closed subgroups. Subscripts are used to emphasise that a subgroup is closed respectively open, as in $H \leq_{\mathrm{c}} G$ respectively $H \leq_{\mathrm{o}} G$. We use left-normed commutators, e.g., $[x, y, z] = [[x, y], z]$.

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2. Preliminaries

2.1. Let G be a finitely generated pro-p group. We consider four natural filtration series on G. The *p-power series* of G is given by

$$\mathcal{P}\colon G^{p^i} = \langle x^{p^i} \mid x \in G \rangle, \quad i \in \mathbb{N}_0.$$

The *lower p-series* (or lower p-central series) of G is given recursively by

$$\mathcal{L}\colon P_1(G) = G, \quad \text{and} \quad P_i(G) = P_{i-1}(G)^p\,[P_{i-1}(G), G] \quad \text{for } i \geq 2,$$

while the *Frattini series* of G is given recursively by

$$\mathcal{F}\colon \Phi_0(G) = G, \quad \text{and} \quad \Phi_i(G) = \Phi_{i-1}(G)^p\,[\Phi_{i-1}(G), \Phi_{i-1}(G)] \quad \text{for } i \geq 1.$$

The (modular) *dimension subgroup series* (or Jennings series or Zassenhaus series) of G can be defined recursively by

$$\mathcal{D}\colon D_1(G) = G, \quad \text{and} \quad D_i(G) = D_{\lceil i/p \rceil}(G)^p \prod_{1 \leq j < i} [D_j(G), D_{i-j}(G)] \quad \text{for } i \geq 2.$$

As a default we set $P_0(G) = D_0(G) = G$.

2.2. Next, we collect auxiliary results to detect Hausdorff dimensions of closed subgroups of pro-p groups. For a countably based infinite pro-p group G, equipped with a filtration series $\mathcal{S}\colon G = G_0 \supseteq G_1 \supseteq \ldots$, and a closed subgroup $H \leq_\mathrm{c} G$ we say that H has *strong Hausdorff dimension in G with respect to $\mathcal{S}$* if

$$\operatorname{hdim}_G^{\mathcal{S}}(H) = \lim_{i \to \infty} \frac{\log_p |HG_i : G_i|}{\log_p |G : G_i|}$$

is given by a proper limit.

The first lemma is an easy variation of [4, Lem. 5.3] and we omit the proof.

Lemma 2.1. *Let G be a countably based infinite pro-p group with closed subgroups $K \leq_\mathrm{c} H \leq_\mathrm{c} G$. Let $\mathcal{S}\colon G = G_0 \supseteq G_1 \supseteq \ldots$ be a filtration series of G and write $\mathcal{S}|_H\colon H = H_0 \supseteq H_1 \supseteq \ldots$, with $H_i = H \cap G_i$ for $i \in \mathbb{N}_0$, for the induced filtration series of H. If K has strong Hausdorff dimension in H with respect to $\mathcal{S}|_H$, then*

$$\operatorname{hdim}_G^{\mathcal{S}}(K) = \operatorname{hdim}_G^{\mathcal{S}}(H) \cdot \operatorname{hdim}_H^{\mathcal{S}|_H}(K).$$

Lemma 2.2. *Let G be a countably based infinite pro-p group with closed subgroups $N \trianglelefteq_\mathrm{c} G$ and $H \leq_\mathrm{c} G$. Let $\mathcal{S}\colon G = G_0 \supseteq G_1 \supseteq \ldots$ be a filtration series of G, and consider the induced filtration series of N and G/N defined by*

$$\mathcal{S}|_N\colon G_i \cap N,\ i \in \mathbb{N}_0, \qquad \text{and} \qquad \mathcal{S}|_{G/N}\colon G_iN/N,\ i \in \mathbb{N}_0.$$

Suppose that N has strong Hausdorff dimension $\xi = \operatorname{hdim}_G^{\mathcal{S}}(N)$ in G, with respect to $\mathcal{S}$. Then we have

$$\operatorname{hdim}_G^{\mathcal{S}}(H) \geq (1-\xi)\ \operatorname{hdim}_{G/N}^{\mathcal{S}|_{G/N}}(HN/N) + \xi\ \varliminf_{i \to \infty} \frac{\log_p |HG_i \cap N : G_i \cap N|}{\log_p |N : G_i \cap N|} \qquad (*)$$

$$\geq (1-\xi)\ \operatorname{hdim}_{G/N}^{\mathcal{S}|_{G/N}}(HN/N) + \xi\ \operatorname{hdim}_N^{\mathcal{S}|_N}(H \cap N). \qquad (**)$$

Moreover, equality holds in $(*)$, if HN/N has strong Hausdorff dimension in G/N with respect to $\mathcal{S}|_{G/N}$ or if the lower limit on the right-hand side is actually a limit. Similarly, equality holds in $(**)$ if

(i) *$H \cap N \leq_\mathrm{o} N$ is an open subgroup or*
(ii) *$G_iN = (G_i \cap H)N$, for all sufficiently large $i \in \mathbb{N}$.*

Proof. We observe that

$$\operatorname{hdim}_G^{\mathcal{S}}(H) = \varliminf_{i\to\infty} \Bigg(\underbrace{\frac{\log_p|G:NG_i|}{\log_p|G:G_i|}}_{\to 1-\xi \text{ as } i\to\infty} \frac{\log_p|HG_iN:G_iN|}{\log_p|G:G_iN|} + \underbrace{\frac{\log_p|NG_i:G_i|}{\log_p|G:G_i|}}_{\to \xi \text{ as } i\to\infty} \frac{\log_p|HG_i\cap NG_i:G_i|}{\log_p|NG_i:G_i|} \Bigg)$$

and that, for each $i \in \mathbb{N}_0$,

$$\frac{\log_p|HG_i\cap NG_i:G_i|}{\log_p|NG_i:G_i|} = \frac{\log_p|HG_i\cap N:G_i\cap N|}{\log_p|N:G_i\cap N|}.$$

Finally,

$$\log_p|HG_i\cap N:G_i\cap N| \geq \log_p|(H\cap N)(G_i\cap N):G_i\cap N|$$

and, if condition (i) or (ii) holds, the difference between the two terms is bounded by a constant that is independent of $i \in \mathbb{N}_0$. ☐

Lemma 2.3. *Let $Z \cong C_p^{\aleph_0}$ be a countably based infinite elementary abelian pro-p group, equipped with a filtration series $\mathcal{S}$. Then, for every $\eta \in [0,1]$, there exists a closed subgroup $K \leq_c Z$ with strong Hausdorff dimension η in Z with respect to $\mathcal{S}$.*

Proof. Write $\mathcal{S}\colon Z = Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \ldots$ and let $\eta \in [0,1]$. For $i \in \mathbb{N}$, we have $Z_{i-1}/Z_i \cong C_p^{d_i}$ for non-negative integers d_i.
Claim: There exist non-negative integers $e_1, e_2, \ldots$ such that, for each $i \in \mathbb{N}$, we have $0 \leq e_i \leq d_i$ and

$$e_1 + \ldots + e_i = \lceil \eta(d_1 + \ldots + d_i) \rceil.$$

Indeed, with $e_1 = \lceil \eta d_1 \rceil$ the statement holds true for $i = 1$. Now, let $i \geq 2$ and suppose that $e_1 + \ldots + e_{i-1} = \lceil \eta(d_1 + \ldots + d_{i-1}) \rceil$. Then

$$\lceil \eta(d_1 + \ldots + d_{i-1}) \rceil \leq \lceil \eta(d_1 + \ldots + d_i) \rceil \leq \lceil \eta(d_1 + \ldots + d_{i-1}) \rceil + d_i$$

and thus we may set

$$e_i = \lceil \eta(d_1 + \ldots + d_i) \rceil - (e_1 + \ldots + e_{i-1}),$$

to satisfy the statement for i. The claim is proved.

For all sufficiently large $i \in \mathbb{N}$ we have $d_1 + \ldots + d_i > 0$ and

$$\eta \leq \frac{e_1 + \ldots + e_i}{d_1 + \ldots + d_i} \leq \eta + \frac{1}{d_1 + \ldots + d_i}.$$

With these preparations, it suffices to display a subgroup $K \leq_c Z$ such that

$$\log_p |KZ_i : Z_i| = e_1 + \ldots + e_i.$$

For this purpose, we write

$$Z = \langle z_{1,1}, \ldots, z_{1,d_1},\ \ z_{2,1}, \ldots, z_{2,d_2},\ \ \ldots\ ,\ \ z_{i,1}, \ldots, z_{i,d_i},\ \ \ldots \rangle$$

such that $Z_{i-1} = \langle z_{i,1}, \ldots, z_{i,d_i} \rangle Z_i$ for each $i \in \mathbb{N}$. Then we set

$$K = \langle z_{1,1}, \ldots, z_{1,e_1},\ \ z_{2,1}, \ldots, z_{2,e_2},\ \ \ldots\ ,\ \ z_{i,1}, \ldots, z_{i,e_i},\ \ \ldots \rangle.$$ ☐

Corollary 2.4. *Let G be a countably based pro-p group, equipped with a filtration series $\mathcal{S}$, and let $N \trianglelefteq_{\mathrm{c}} H \leq_{\mathrm{c}} G$ such that $H/N \cong C_p^{\aleph_0}$. Set $\xi = \mathrm{hdim}_G^{\mathcal{S}}(N)$ and $\eta = \mathrm{hdim}_G^{\mathcal{S}}(H)$. If N or H has strong Hausdorff dimension in G with respect to $\mathcal{S}$, then $[\xi, \eta] \subseteq \mathrm{hspec}^{\mathcal{S}}(G)$.*

Proof. If N has strong Hausdorff dimension, we apply Lemmata 2.1, 2.2 and 2.3. If H has strong Hausdorff dimension the claim follows from [4, Thm. 5.4]. □

2.3. For convenience we recall two standard commutator collection formulae.

Proposition 2.5. *Let $G = \langle a, b\rangle$ be a finite p-group, and let $r \in \mathbb{N}$. For $u, v \in G$ let $K(u, v)$ denote the normal closure in G of* (i) *all commutators in $\{u, v\}$ of weight at least p^r that have weight at least 2 in v, together with* (ii) *the p^{r-s+1}th powers of all commutators in $\{u, v\}$ of weight less than p^s and of weight at least 2 in v for $1 \leq s \leq r$. Then*

$$(ab)^{p^r} \equiv_{K(a,b)} a^{p^r} b^{p^r} [b,a]^{\binom{p^r}{2}} [b,a,a]^{\binom{p^r}{3}} \cdots [b,a,\overset{p^r-2}{\ldots},a]^{\binom{p^r}{p^r-1}} [b,a,\overset{p^r-1}{\ldots},a], \quad (2.1)$$

$$[a^{p^r}, b] \equiv_{K(a,[a,b])} [a,b]^{p^r} [a,b,a]^{\binom{p^r}{2}} \cdots [a,b,a,\overset{p^r-2}{\ldots},a]^{\binom{p^r}{p^r-1}} [a,b,a,\overset{p^r-1}{\ldots},a]. \quad (2.2)$$

Remark. *Under the standing assumption $p \geq 3$ and the extra assumptions*

$$\gamma_2(G)^p = 1 \qquad \text{and} \qquad [\gamma_2(G), \gamma_2(G)] \subseteq Z(G),$$

the congruences (2.1) *and* (2.2) *simplify to*

$$(ab)^{p^r} \equiv_{L(a,b)} a^{p^r} b^{p^r} [b,a,\overset{p^r-1}{\ldots},a] \quad \text{and} \quad [a^{p^r}, b] \equiv_{M(a,b)} [a,b,a,\overset{p^r-1}{\ldots},a], \quad (2.3)$$

where $L(a,b)$ denotes the normal closure in G of all commutators in $\{a,b\}$ of weight at least p^r that have weight at least 2 in b and $M(a,b)$ denotes the normal closure in G of all commutators $[[b,a,\overset{i}{\ldots},a],[b,a,\overset{j}{\ldots},a]]$ with $i + j \geq p^r$.

The general result is recorded (in a slighter stronger form) in [5, Prop. 1.1.32]; we remark that (2.2) follows directly from (2.1), due to the identity $[a^{p^r}, b] = a^{-p^r}(a[a,b])^{p^r}$. The first congruence in (2.3) follows directly from (2.1); the second congruence in (2.3) is derived from (2.2) by standard commutator manipulations.

2.4. Now we describe, for $k \in \mathbb{N}$, the lower central series, the lower p-series and the Frattini series of the finite wreath product

$$W_k = \langle x, y\rangle = \langle x\rangle \ltimes \langle y, y^x, \ldots, y^{x^{p^k-1}}\rangle \cong C_p \wr C_{p^k}$$

with top group $\langle x\rangle \cong C_{p^k}$ and base group $\langle y, y^x, \ldots, y^{x^{p^k-1}}\rangle \cong C_p^{p^k}$.

Proposition 2.6. *For $k \in \mathbb{N}$, the finite wreath product W_k defined above is nilpotent of class p^k and $W_k^{p^k} = \langle y y^x y^{x^2} \cdots y^{x^{p^k-1}}\rangle \cong C_p$.*

(1) *The lower central series of W_k satisfies*

$$W_k = \gamma_1(W_k) = \langle x, y\rangle\, \gamma_2(W_k) \qquad \text{with} \quad W_k/\gamma_2(W_k) \cong C_{p^k} \times C_p,$$

$$\gamma_i(W_k) = \langle [y,x,\overset{i-1}{\ldots},x]\rangle\, \gamma_{i+1}(W_k) \quad \text{with} \quad \gamma_i(W_k)/\gamma_{i+1}(W_k) \cong C_p \text{ for } 2 \leq i \leq p^k.$$

(2) *The lower p-series of W_k has length p^k; it satisfies, for $1 \leq i \leq k$,*

$$P_i(W_k) = \langle x^{p^{i-1}}, [y,x,\overset{i-1}{\ldots},x]\rangle\, P_{i+1}(W_k) \quad \text{with} \quad P_i(W_k)/P_{i+1}(W_k) \cong C_p \times C_p,$$

and, for $k < i \leq p^k$,

$$P_i(W_k) = \langle [y,x,\overset{i-1}{\ldots},x]\rangle\, P_{i+1}(W_k) \qquad \text{with} \quad P_i(W_k)/P_{i+1}(W_k) \cong C_p.$$

(3) *The Frattini series of W_k has length $k+1$; it satisfies, for $0 \le i < k$,*

$$\Phi_i(W_k) = \langle x^{p^i} \rangle \gamma_{\frac{p^i-1}{p-1}+1}(W_k) \quad \textit{with} \quad \Phi_i(W_k)/\Phi_{i+1}(W_k) \cong C_p \times \overset{p^i+1}{\cdots} \times C_p,$$

$$\Phi_k(W_k) = \gamma_{\frac{p^k-1}{p-1}+1}(W_k) \qquad \textit{with} \quad \Phi_k(W_k)/\Phi_{k+1}(W_k) \cong C_p \times \overset{\frac{p^{k+1}-2p^k+1}{p-1}}{\cdots} \times C_p.$$

(4) *The dimension subgroup series of W_k has length p^k; in particular, it satisfies, for $p^{k-1}+1 \le i \le p^k$,*

$$D_i(W_k) = \gamma_i(W_k) = \langle [y, x, \overset{i-1}{\ldots}, x] \rangle \, D_{i+1}(W_k) \quad \textit{with} \quad D_i(W_k)/D_{i+1}(W_k) \cong C_p.$$

Proof. The assertions are well known and easy to verify from the concrete realisation of W_k as a semidirect product

$$W_k \cong \langle 1+t \rangle / \langle (1+t)^{p^k} \rangle \ltimes \mathbb{F}_p[t]/t^{p^k}\mathbb{F}_p[t] \tag{2.4}$$

in terms of polynomials over the finite field $\mathbb{F}_p$: here y^{x^i} corresponds to $(1+t)^i$ modulo $t^{p^k}\mathbb{F}_p[t]$, and it is easy to describe all normal subgroups. In particular the normal subgroups of W_k contained in the base group form a descending chain, corresponding to the groups $t^{i-1}\mathbb{F}_p[t]/t^{p^k}\mathbb{F}_p[t]$, $1 \le i \le p^k+1$.

For $0 \le m < k$ and $z \in \langle y, y^x, \ldots, y^{x^{p^k-1}} \rangle$ the element

$$(x^{p^m} z)^{p^k} = (x^{p^m})^{p^k} z^{x^{(p^k-1)p^m}} \cdots z^{x^{p^m}} z = z^{x^{(p^k-1)p^m}} \cdots z^{x^{p^m}} z$$

corresponds in $\mathbb{F}_p[t]/t^{p^k}\mathbb{F}_p[t]$ to a multiple of

$$\sum_{i=0}^{p^k-1} (1+t)^{ip^m} = \sum_{i=0}^{p^k-1} (1+t^{p^m})^i = \frac{(1+t^{p^m})^{p^k}-1}{(1+t^{p^m})-1} = t^{(p^k-1)p^m};$$

this shows that $W_k^{p^k} = \langle y y^x y^{x^2} \cdots y^{x^{p^k-1}} \rangle \cong C_p$.

Clearly, $\gamma_1(W_k) = W_k$. For $2 \le i \le p^k+1$, the group $\gamma_i(W_k)$ corresponds to the subgroup $t^{i-1}\mathbb{F}_p[t]/t^{p^k}\mathbb{F}_p[t]$ of the base group. In particular, W_k has nilpotency class p^k. For $1 \le i \le k$, we have $P_i(W_k) = \langle x^{p^{i-1}} \rangle \gamma_i(W_k)$, while for $k < i \le p^k$ we get $P_i(W_k) = \gamma_i(W_k)$. For $0 \le i \le k$ a simple induction shows that the group $\Phi_i(W_k)$ is the normal closure in W_k of the two elements

$$x^{p^i} \qquad \text{and} \qquad [y, x, x^p, x^{p^2}, \ldots, x^{p^{i-1}}] = [y, x, \overset{\frac{p^i-1}{p-1}}{\ldots}, x];$$

the intersection of $\Phi_i(W_k)$ with the base group corresponds to $t^{\frac{p^i-1}{p-1}}\mathbb{F}_p[t]/t^{p^k}\mathbb{F}_p[t]$. Thus $\Phi_i(W_k) = \langle x^{p^i} \rangle \gamma_{\frac{p^i-1}{p-1}+1}(W_k)$. In particular, $\Phi_k(W_k)$ is elementary abelian and $\Phi_i(W_k) = 1$ for $i > k$. Finally, for $i \ge p^{k-1}+1$, we use [2, Thm. 11.2] to deduce that $D_i(W_k) = \gamma_i(W_k)$. □

The structural results for the finite wreath products W_k transfer naturally to the inverse limit $W \cong \varprojlim_k W_k$, i.e., the pro-$p$ wreath product

$$W = \langle x, y \rangle = \langle x \rangle \ltimes B \cong C_p \mathbin{\hat{\wr}} \mathbb{Z}_p \tag{2.5}$$

with top group $\langle x \rangle \cong \mathbb{Z}_p$ and base group $B = \prod_{i \in \mathbb{Z}} \langle y^{x^i} \rangle \cong C_p^{\aleph_0}$. Compatible with (2.4), the group W has a concrete realisation as a semidirect product

$$W \cong \langle 1+t \rangle \ltimes \mathbb{F}_p[\![t]\!], \quad \text{where } \langle 1+t \rangle \le_{\mathrm{c}} \mathbb{F}_p[\![t]\!]^*, \tag{2.6}$$

in terms of formal power series over the finite field $\mathbb{F}_p$. We record the following lemma on closed normal subgroups of W.

Lemma 2.7. *Let $W = \langle x, y\rangle \cong C_p \,\hat{\wr}\, \mathbb{Z}_p$ with base group B as above, and let $1 \neq K \trianglelefteq_{\mathrm{c}} W$ be a non-trivial closed normal subgroup. Then either K is open in W or K is open in B; in particular, $K \cap B \le_{\mathrm{o}} B$ and $|K \cap B : [K \cap B, W]| = p$.*

Proof. The lower central series of W is well known and easy to compute: $\gamma_1(W) = W$ and $\gamma_i(W) = B_{i-1}$ for $i \geq 2$, where $B = B_0 \geq B_1 \geq B_2 \geq \ldots$ with $B_{i-1} = \langle [y, x, \overset{i-1}{\ldots}, x]\rangle B_i$ and $|B_{i-1} : B_i| = p$; in other words, $\langle x\rangle$ acts uniserially on B; compare Proposition 2.6.

It follows that $1 \neq K \cap B = B_i$ for some non-negative integer i, hence $K \cap B \le_{\mathrm{o}} B$ and $|K \cap B : [K \cap B, W]| = |B_i : B_{i+1}| = p$. Suppose now that $K \not\subseteq B$. Then there exists $x^m z \in K$ with $m \in \mathbb{N}$ and $z \in B$. We may assume that $m = p^k$ is a p-power. Then $\langle x^{p^k} z\rangle \ltimes B \cong \langle x\rangle \ltimes (B \times \overset{p^k}{\ldots} \times B)$, where $x^{p^k} z$ maps to x and, on the right-hand side, x acts diagonally and in each coordinate according to the original action in W. Hence we may assume that $x \in K$. Now the description of the lower central series of W yields $\langle x\rangle B_1 \le_{\mathrm{c}} K$ and thus $K \le_{\mathrm{o}} W$. ☐

From Proposition 2.6 and Lemma 2.7 we deduce the following; cf. [3, Ch. VIII.7].

Corollary 2.8. *The normal Hausdorff spectrum of the pro-p group $W = C_p \,\hat{\wr}\, \mathbb{Z}_p$ with respect to the standard filtration series $\mathcal{P}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{L}$ respectively, satisfies:*

$$\mathrm{hspec}^{\mathcal{P}}_{\trianglelefteq}(W) = \mathrm{hspec}^{\mathcal{D}}_{\trianglelefteq}(W) = \mathrm{hspec}^{\mathcal{F}}_{\trianglelefteq}(W) = \{0, 1\} \quad \textit{and} \quad \mathrm{hspec}^{\mathcal{L}}_{\trianglelefteq}(W) = \{0, 1/2, 1\}.$$

The next result is well known (and not difficult to prove directly); compare [9, Cor. 12.5.10]. It gives a first indication that Theorem 1.1 is at least plausible.

Proposition 2.9. *The pro-p group $W = C_p \,\hat{\wr}\, \mathbb{Z}_p$ is not finitely presented.*

The final result in this section concerns the *finitely generated Hausdorff spectrum* of the pro-p group $W = C_p \,\hat{\wr}\, \mathbb{Z}_p$, with respect to a standard filtration series $\mathcal{S}$; it is defined as

$$\mathrm{hspec}^{\mathcal{S}}_{\mathrm{fg}}(W) = \{\mathrm{hdim}^{\mathcal{S}}_W(H) \mid H \le_{\mathrm{c}} W \text{ and } H \text{ finitely generated}\}$$

and reflects the range of Hausdorff dimensions of (topologically) finitely generated subgroups; compare [7, §4.7].

Theorem 2.10. *With respect to the standard filtration series $\mathcal{P}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{L}$ respectively, the pro-p group $W = C_p \,\hat{\wr}\, \mathbb{Z}_p$ satisfies:*

$$\mathrm{hspec}^{\mathcal{P}}_{\mathrm{fg}}(W) = \mathrm{hspec}^{\mathcal{D}}_{\mathrm{fg}}(W) = \mathrm{hspec}^{\mathcal{F}}_{\mathrm{fg}}(W) = \{m/p^n \mid n \in \mathbb{N}_0, 0 \le m \le p^n\},$$
$$\mathrm{hspec}^{\mathcal{L}}_{\mathrm{fg}}(W) = \{0\} \cup \{1/2 + m/2p^n \mid n \in \mathbb{N}_0, 0 \le m \le p^n\}.$$

Proof. As above, let B denote the base group of the wreath product $W = \langle x, y\rangle$. Let $\mathcal{S} \in \{\mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L}\}$, and let K be a finitely generated subgroup of W.

If $K \subseteq B$ then K is finite and $\mathrm{hdim}^{\mathcal{S}}_W(K) = 0$. Now suppose that $K \not\subseteq B$; in the proof below we will no longer use that K is finitely generated, but it will become clear that this is automatically so. Write $K = \langle x^{p^n} z\rangle \ltimes M$, where $n \in \mathbb{N}_0$, $z \in B$ and $M = K \cap B$. Let $B = B_0 \geq B_1 \geq \ldots$ be the filtration corresponding to $\mathbb{F}_p[\![t]\!] \geq t\mathbb{F}_p[\![t]\!] \geq \ldots$ under (2.6), as in the proof of Lemma 2.7. We set

$$J = \{j \in \mathbb{N}_0 \mid (M \cap B_j) \not\subseteq B_{j+1}\} \quad \text{and} \quad J_0 = \{j + p^n\mathbb{Z} \mid j \in J\} \subseteq \mathbb{Z}/p^n\mathbb{Z}.$$

Under the isomorphism (2.6), we may regard M as an $\mathbb{F}_p[\![t^{p^n}]\!]$-submodule of $\mathbb{F}_p[\![t]\!]$. Hence $J + p^n\mathbb{N}_0 \subseteq J$ and

$$\lim_{i\to\infty} \frac{\log_p|(K \cap B)B_i : B_i|}{\log_p|B : B_i|} = \frac{|J_0|}{p^n}.$$

From Proposition 2.6 it is easily seen that B has strong Hausdorff dimension

$$\mathrm{hdim}_W^{\mathcal{P}}(B) = \mathrm{hdim}_W^{\mathcal{D}}(B) = \mathrm{hdim}_W^{\mathcal{F}}(B) = 1 \quad \text{and} \quad \mathrm{hdim}_W^{\mathcal{L}}(B) = 1/2;$$

compare Corollary 2.8. Using Lemma 2.2, we deduce that

$$\mathrm{hdim}_W^{\mathcal{S}}(K) = (1 - \mathrm{hdim}_W^{\mathcal{S}}(B)) + \mathrm{hdim}_W^{\mathcal{S}}(B)\frac{|J_0|}{p^n}$$

lies in the desired range; in fact, the argument even shows that K has strong Hausdorff dimension.

Conversely, our analysis above shows that, for $n \in \mathbb{N}_0$ and $0 \le m \le p^n$, the subgroup $K_{n,m} = \langle x^{p^n}, [y,x], [y,x,x], \ldots, [y,x,\overset{m}{\ldots},x]\rangle$ has Hausdorff dimension

$$\mathrm{hdim}_W^{\mathcal{S}}(K_{n,m}) = \begin{cases} m/p^n & \text{if } \mathcal{S} \in \{\mathcal{P},\mathcal{D},\mathcal{F}\}, \\ 1/2 + m/2p^n & \text{if } \mathcal{S} = \mathcal{L}. \end{cases} \qquad \square$$

The next corollary answers a question raised in [3, VIII.7.2]; it was shown there that $[0, 1/2] \subseteq \mathrm{hspec}^{\mathcal{L}}(W)$, while $(1/2, 1) \cap \mathrm{hspec}^{\mathcal{L}}(W)$ remained undetermined.

Corollary 2.11. *The Hausdorff spectrum of the pro-p group $W = C_p \,\hat{\wr}\, \mathbb{Z}_p$ with respect to the lower p-series $\mathcal{L}$ is*

$$\mathrm{hspec}^{\mathcal{L}}(W) = [0, 1/2] \cup \{1/2 + m/2p^n \mid n \in \mathbb{N}_0, 1 \le m \le p^n - 1\} \cup \{1\}.$$

Furthermore, every subgroup $K \le_{\mathrm{c}} W$ with $\mathrm{hdim}_W^{\mathcal{L}}(K) > 1/2$ has strong Hausdorff dimension in W, with respect to $\mathcal{L}$.

Proof. The subgroups contained in the base group B of W yield $[0, 1/2]$ as part of the Hausdorff spectrum; cf. Lemma 2.3. The proof of Theorem 2.10 shows that the subgroups not contained in B yield the remaining part of the claimed spectrum and that each of them has strong Hausdorff dimension in W. $\square$

3. An explicit presentation for the pro-p group G and a description of its finite quotients G_k for $k \in \mathbb{N}$

Recall that p is an odd prime. As indicated in the paragraph before Theorem 1.1, we consider the pro-p group $G = F/N$, where

- $F = \langle x, y\rangle$ is a free pro-p group and
- $N = [R, F]R^p \trianglelefteq_{\mathrm{c}} F$ for the kernel $R \trianglelefteq_{\mathrm{c}} F$ of the presentation $\pi\colon F \to W$ sending x, y to the generators of the same name in (2.5).

By producing generators for R and N as closed normal subgroups of F we obtain explicit presentations for the pro-p groups W and G.

It is convenient to write $y_i = y^{x^i}$ for $i \in \mathbb{Z}$. Setting

$$R_k = \langle \{x^{p^k}, y^p\} \cup \{[y_0, y_i] \mid 1 \le i \le \tfrac{p^k-1}{2}\}\rangle^F \trianglelefteq_{\mathrm{o}} F \tag{3.1}$$

for $k \in \mathbb{N}$, we obtain a descending chain of open normal subgroups

$$F \supseteq R_1 \supseteq R_2 \supseteq \ldots \tag{3.2}$$

with quotient groups $F/R_k \cong W_k \cong C_p \wr C_{p^k}$. We put

$$R = \langle \{y^p\} \cup \{[y_0, y_i] \mid i \in \mathbb{N}\}\rangle^F \trianglelefteq_{\mathrm{c}} F,$$

and observe that $R_k = \langle x^{p^k}\rangle^F R$ for each $k \in \mathbb{N}$. Since $x^{p^k} \to 1$ as $k \to \infty$, this yields $R = \bigcap_{k\in\mathbb{N}} R_k$ and thus $F/R \cong W \cong C_p \,\hat{\wr}\, \mathbb{Z}_p$. With hindsight there is no harm in taking $W_k = F/R_k$ for $k \in \mathbb{N}$ and $W = F/R$.

Setting $N_k = [R_k, F]R_k^p$ for $k \in \mathbb{N}$, we observe that

$$N_k = \langle \{x^{p^{k+1}}, y^{p^2}, [x^{p^k}, y], [y^p, x]\} \cup \{[y_0, y_i]^p \mid 1 \le i \le \tfrac{p^k-1}{2}\} \\ \cup \{[y_0, y_i, x] \mid 1 \le i \le \tfrac{p^k-1}{2}\} \cup \{[y_0, y_i, y] \mid 1 \le i \le \tfrac{p^k-1}{2}\}\rangle^F \trianglelefteq_{\mathrm{o}} F,$$

and as in (3.2) we obtain a descending chain $F \supseteq N_1 \supseteq N_2 \supseteq \ldots$ of open normal subgroups. Moreover, it follows that $\bigcap_{k\in\mathbb{N}} N_k \supseteq [R, F]R^p = N$. On the other hand, if $z \notin N$ then there exists an open normal subgroup $K \trianglelefteq_{\mathrm{o}} F$ and $k \in \mathbb{N}$ such that $z \notin NK = [R_k, F]R_k^p K$, hence $z \notin N_k$. Thus we conclude that

$$\bigcap\nolimits_{k\in\mathbb{N}} N_k = [R, F]R^p = N.$$

Consequently, $G = F/N \cong \varprojlim G_k$, where

$$G_k = F/N_k \cong \langle x, y \mid \underline{x^{p^{k+1}}},\ y^{p^2},\ \underline{[x^{p^k}, y]},\ [y^p, x]; \\ [y_0, y_i]^p,\ [y_0, y_i, x],\ [y_0, y_i, y] \quad \text{for } 1 \le i \le \tfrac{p^k-1}{2}\rangle \tag{3.3}$$

for $k \in \mathbb{N}$, and

$$G \cong \langle x, y \mid y^{p^2},\ [y^p, x];\ [y_0, y_i]^p,\ [y_0, y_i, x],\ [y_0, y_i, y] \quad \text{for } i \in \mathbb{N}\rangle \tag{3.4}$$

is a presentation of G as a pro-p group. Indeed,

$$\widetilde{N} = \langle \{y^{p^2}, [y^p, x]\} \cup \{[y_0, y_i]^p, [y_0, y_i, x], [y_0, y_i, y] \mid i \in \mathbb{N}\}\rangle^F \trianglelefteq_{\mathrm{c}} F$$

satisfies, for each $k \in \mathbb{N}$,

$$N_k = \langle x^{p^{k+1}}, [x^{p^k}, y]\rangle^F\, \widetilde{N},$$

where $x^{p^{k+1}}, [x^{p^k}, y] \to 1$ as $k \to \infty$. This yields $\widetilde{N} = \bigcap_{k\in\mathbb{N}} N_k = N$. To facilitate later use, we have underlined the two relations in (3.3) that do not yet occur in (3.4).

To summarise and supplement some of the notation introduced above, we define

$$Y = \langle y_i \mid i \in \mathbb{Z}\rangle R \trianglelefteq_{\mathrm{c}} F, \qquad H = Y/N \trianglelefteq_{\mathrm{c}} G, \qquad Z = R/N \trianglelefteq_{\mathrm{c}} G.$$

Similarly for $k \in \mathbb{N}$ we set

$$Y_k = \langle y_i \mid i \in \mathbb{Z}\rangle R_k \trianglelefteq_{\mathrm{o}} F, \qquad H_k = Y_k/N_k \trianglelefteq G_k, \qquad Z_k = R_k/N_k \trianglelefteq G_k.$$

Diagrammatically, we have:

$$\left.\begin{array}{ccc} F & \twoheadrightarrow & G \\ | & & | \\ Y & \twoheadrightarrow & H \\ | & & | \\ R & \twoheadrightarrow & Z \end{array}\right\} G/Z \cong W \qquad\qquad W_k \cong G_k/Z_k \left\{\begin{array}{ccc} G_k & \twoheadleftarrow & F \\ | & & | \\ H_k & \twoheadleftarrow & Y_k \\ | & & | \\ Z_k & \twoheadleftarrow & R_k \end{array}\right.$$
$$\begin{array}{ccc} | & & | \\ N & \twoheadrightarrow & 1 \end{array} \qquad\qquad\qquad\qquad\qquad\qquad \begin{array}{ccc} | & & | \\ 1 & \twoheadleftarrow & N_k \end{array}$$

Lemma 3.1. *The centre of G is $Z(G) = Z$, and $Z_k \le Z(G_k)$ for $k \in \mathbb{N}$.*

Proof. By construction, $Z \le Z(G)$ and $Z_k \le Z(G_k)$ for $k \in \mathbb{N}$. From (2.6) we see that $G/Z \cong W$ has trivial centre. Therefore $Z = Z(G)$. ☐

In fact, $Z_k \lneqq Z(G_k)$ for $k \in \mathbb{N}$; see Lemma 5.3 below.

4. General description of the normal Hausdorff spectrum of the pro-p group G and its finite direct powers

We continue to use the notation set up in Section 3 to study the pro-p group G and its finite direct powers.

Proposition 4.1. *Let $K \trianglelefteq_{\mathrm{c}} G$ be a closed normal subgroup such that $K \not\subseteq Z$. Then either K is open in H or K is open in G; in particular, $K \cap H \le_{\mathrm{o}} H$. Furthermore, $[K \cap H, G] \le_{\mathrm{o}} H$.*

Proof. Lemma 2.7 shows: $KZ \cap H \le_{\mathrm{o}} H$; hence it suffices to prove $K \cap Z \le_{\mathrm{o}} Z$. Choose $\hat{y}_1, \hat{y}_2, \ldots \in H$, converging to 1 modulo Z, and $m \in \mathbb{N}$ such that (the images of) $\hat{y}_1, \hat{y}_2, \ldots$ (modulo Z) yield a basis for the elementary abelian pro-p group H/Z and $\hat{y}_{m+1}, \hat{y}_{m+2}, \ldots$ generate $KZ \cap H$ modulo Z.

Recall that Z is central in G and of exponent p. Thus $K \cap Z$ contains $\hat{y}_i^{\,p}$ and $[\hat{y}_i, \hat{y}_j]$ for all $i, j \in \mathbb{N}$ with $i > m$. Hence the finite set

$$\{\hat{y}_i^{\,p} \mid 1 \le i \le m\} \cup \{[\hat{y}_i, \hat{y}_j] \mid 1 \le i \le j \le m\}$$

generates the elementary abelian group Z modulo $K \cap Z$, and $K \cap Z \le_{\mathrm{o}} Z$.

Finally, Lemma 2.7 implies that $[K \cap H, G] \not\subseteq Z$. Hence $[K \cap H, G] \le_{\mathrm{o}} H$. □

From Proposition 4.1, Lemma 3.1 and Lemmata 2.1 and 2.3 we deduce the general shape of the normal Hausdorff spectrum of G.

Corollary 4.2. *Let $\mathcal{S}$ be an arbitrary filtration series of G. Then the normal Hausdorff spectrum of G has the form*

$$\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(G) = [0, \xi] \cup \{\eta\} \cup \{1\},$$

where $\xi = \mathrm{hdim}^{\mathcal{S}}_G(Z)$ and $\eta = \mathrm{hdim}^{\mathcal{S}}_G(H)$.

More generally we obtain a description of the normal Hausdorff spectrum of finite direct powers $G^{(m)} = G \times \overset{m}{\cdots} \times G$ of G, with respect to suitable 'product filtration series'. For any filtration series $\mathcal{S}\colon G = S_0 \supseteq S_1 \ldots$ of G we consider the naturally induced product filtration series on $G^{(m)}$ given by

$$\mathcal{S}^{(m)}\colon G^{(m)} = G \times \overset{m}{\cdots} \times G \supseteq S_1 \times \overset{m}{\cdots} \times S_1 \supseteq S_2 \times \overset{m}{\cdots} \times S_2 \supseteq \ldots.$$

For a standard filtration series $\mathcal{S} \in \{\mathcal{P}, \mathcal{L}, \mathcal{F}, \mathcal{D}\}$ on G the product filtration series $\mathcal{S}^{(m)}$ is actually the corresponding standard filtration series on $G^{(m)}$.

Corollary 4.3. *Let $m \in \mathbb{N}$, and let $K \trianglelefteq_{\mathrm{c}} G^{(m)}$. For $1 \le j \le m$, let $\pi_j\colon G^{(m)} \to G$ be the canonical projection onto the jth factor and set*

$$\overline{K}(j) = \begin{cases} Z & \text{if } K\pi_j \subseteq Z, \\ G & \text{otherwise,} \end{cases} \quad \text{and} \quad \underline{K}(j) = \begin{cases} 1 & \text{if } K\pi_j \subseteq Z, \\ H & \text{otherwise.} \end{cases}$$

Then $K \le \prod_{j=1}^m \overline{K}(j)$ and K contains an open normal subgroup of $\prod_{j=1}^m \underline{K}(j)$.

Proof. Observe that

$$[K\pi_1, G] \times \ldots \times [K\pi_m, G] = [K, G^{(m)}] \le K \le K\pi_1 \times \ldots \times K\pi_m.$$

Thus K is contained in $\prod_{j=1}^m \overline{K}(j)$, and it suffices to show that $[K\pi_j \cap H, G] \le_{\mathrm{o}} H$ for each j with $K\pi_j \not\subseteq Z$. This follows by Proposition 4.1. □

Corollary 4.4. *Let $m \in \mathbb{N}$, and let $\mathcal{S}$ be a filtration series of G such that* $\mathrm{hdim}_G^{\mathcal{S}}(H) = 1$. *Then the normal Hausdorff spectrum of $G^{(m)}$ has the form*

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}^{(m)}}(G^{(m)}) = [0, \xi] \cup \bigcup_{1 \le l \le m-1} \left[{}^{l}\!/\!{}_{m}, {}^{l+(m-l)\xi}\!/\!{}_{m} \right] \cup \{1\},$$

where $\xi = \mathrm{hdim}_G^{\mathcal{S}}(Z)$.

Proof. First let $K \trianglelefteq_{\mathrm{c}} G^{(m)}$, and define $\overline{K}(j), \underline{K}(j)$ for $1 \le j \le m$ as in Corollary 4.3. From $\mathrm{hdim}_G^{\mathcal{S}}(H) = 1$ we deduce that

$$\begin{aligned} {}^{l}\!/\!{}_{m} = \mathrm{hdim}_{G^{(m)}}^{\mathcal{S}^{(m)}} \left(\textstyle\prod_{j=1}^{m} \underline{K}(j) \right) &\le \mathrm{hdim}_{G^{(m)}}^{\mathcal{S}^{(m)}}(K) \\ &\le \mathrm{hdim}_{G^{(m)}}^{\mathcal{S}^{(m)}} \left(\textstyle\prod_{j=1}^{m} \overline{K}(j) \right) = {}^{l}\!/\!{}_{m} + {}^{m-l}\!/\!{}_{m}\, \xi, \end{aligned}$$

where $l = \#\{ j \mid 1 \le j \le m \text{ and } \overline{K}(j) = G \}$.

Conversely, for every $l \in \{0, 1, \ldots, m\}$ and $\beta \in \left[{}^{l}\!/\!{}_{m}, {}^{l+(m-l)\xi}\!/\!{}_{m} \right]$ there is a normal subgroup

$$K_\beta = G \times \overset{l}{\ldots} \times G \times U \times \overset{m-l}{\ldots} \times U \trianglelefteq_{\mathrm{c}} G^{(m)},$$

where $U \le_{\mathrm{c}} Z$ for $l < m$ has $\mathrm{hdim}_G^{\mathcal{S}}(U) = {}^{m}\!/\!{}_{m-l}\,(\beta - {}^{l}\!/\!{}_{m}) \in [0, \xi]$; compare Corollary 4.2. This yields $\beta = \mathrm{hdim}_{G^{(m)}}^{\mathcal{S}^{(m)}}(K_\beta) \in \mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}^{(m)}}(G^{(m)})$. ☐

Corollary 4.4 shows that, once $\mathrm{hdim}_G^{\mathcal{S}}(H) = 1$, the general shape (e.g. the number of connected components) of the normal Hausdorff spectrum $\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}^{(m)}}(G^{(m)})$ depends only on the parameters $\xi = \mathrm{hdim}_G^{\mathcal{S}}(Z)$ and $m \in \mathbb{N}$. For instance, if $\xi < {}^{1}\!/\!{}_{m}$, then $\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}^{(m)}}(G^{(m)})$ is the union of $m+1$ disjoint intervals, whereas for $\xi \ge {}^{1}\!/\!{}_{2}$ we obtain $\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}^{(m)}}(G^{(m)}) = [0, 1 - {}^{(1-\xi)}\!/\!{}_{m}] \cup \{1\}$.

The proof of Theorem 1.1 in Sections 5 and 6 will give $\mathrm{hdim}_G^{\mathcal{S}}(H) = 1$ for the standard filtrations $\mathcal{S} \in \{\mathcal{P}, \mathcal{D}, \mathcal{F}\}$ and $\xi = \mathrm{hdim}_G^{\mathcal{P}}(Z) = \mathrm{hdim}_G^{\mathcal{D}}(Z) = {}^{1}\!/\!{}_{3}$ respectively $\xi = \mathrm{hdim}_G^{\mathcal{F}}(Z) = {}^{1}\!/\!{}_{p+1}$; the assertion for H is already a consequence of [4, Prop. 4.2]. We formulate a taylor-made corollary for these situations.

Corollary 4.5. *Let $m, n \in \mathbb{N}$ with $m \ge \max\{2, n-1\}$ and $n \ge 2$. Let $\mathcal{S}$ be a filtration series of G such that $\mathrm{hdim}_G^{\mathcal{S}}(H) = 1$ and $\mathrm{hdim}_G^{\mathcal{S}}(Z) = {}^{1}\!/\!{}_{n}$. Then*

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}^{(m)}}(G^{(m)}) = \left[0, \tfrac{mn-(n-1)^2}{mn}\right] \cup \bigcup_{m-n+2 \le l \le m-1} \left[\tfrac{l}{m}, \tfrac{m+l(n-1)}{mn} \right] \cup \{1\}$$

consists of n disjoint intervals.

Proof. From Corollary 4.4, we have

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}^{(m)}}(G^{(m)}) = \left[0, {}^{1}\!/\!{}_{n}\right] \cup \bigcup_{1 \le l \le m-1} \left[{}^{l}\!/\!{}_{m}, {}^{m+l(n-1)}\!/\!{}_{mn} \right] \cup \{1\}.$$

For $m - n + 1 \le l \le m - 1$ it is easy to verify that

$$\frac{m + l(n-1)}{mn} < \frac{l+1}{m}.$$

Hence it suffices to show that

$$\left[0, {}^{1}\!/\!{}_{n}\right] \cup \bigcup_{1 \le l \le m-n+1} \left[{}^{l}\!/\!{}_{m}, {}^{m+l(n-1)}\!/\!{}_{mn} \right] = \left[0, \tfrac{mn-(n-1)^2}{mn}\right].$$

For $m = n-1$ this reduces to $\big[0, 1/n\big] = \big[0, \frac{mn-(n-1)^2}{mn}\big]$. Now suppose that $m \geq n$. Then the claim follows from

$$1/m \leq 1/n \qquad \text{and} \qquad l+1/m \leq m+l(n-1)/mn \quad \text{for } 1 \leq l \leq m-n. \qquad \square$$

5. The normal Hausdorff spectrum of G with respect to the p-power series

We continue to use the notation set up in Section 3 and establish that $\xi = \mathrm{hdim}_G^{\mathcal{P}}(Z) = 1/3$ and $\eta = \mathrm{hdim}_G^{\mathcal{P}}(H) = 1$, with respect to the p-power series $\mathcal{P}$. In view of Corollary 4.2 this proves Theorem 1.1 for the p-power series. Indeed, $\mathrm{hdim}_G^{\mathcal{P}}(H) = 1$ is already a consequence of [4, Prop. 4.2]. It remains to show that

$$\mathrm{hdim}_G^{\mathcal{P}}(Z) = \varliminf_{i\to\infty} \frac{\log_p |ZG^{p^i} : G^{p^i}|}{\log_p |G : G^{p^i}|} = 1/3. \tag{5.1}$$

It is convenient to work with the finite quotients G_k, $k \in \mathbb{N}$, introduced in Section 3. Let $k \in \mathbb{N}$. From (3.3) and (3.4) we observe that

$$|G : G^{p^k}| = |G_k : G_k^{p^k}|.$$

Heuristically, $G_k^{p^k}$ is almost trivial (see Proposition 5.2) and the elementary abelian p-group Z_k requires roughly half the number of generators compared to the elementary abelian p-group H_k/Z_k. This suggests that (3.4) should be true. We now work out the details.

First we compute the order of G_k, using the notation from Section 3.

Lemma 5.1. *The logarithmic order of G_k is*

$$\log_p |G_k| = \tfrac{1}{2}(3p^k + 2k + 3).$$

In particular,

$$Z_k = R_k/N_k = \langle \{x^{p^k}, y^p\} \cup \{[y_0, y_i] \mid 1 \leq i \leq \tfrac{p^k-1}{2}\}\rangle N_k/N_k \cong C_p \times \overset{\frac{p^k+3}{2}}{\cdots} \times C_p.$$

Proof. Observe from $F/R_k \cong W_k \cong C_p \wr C_{p^k}$ that

$$\log_p |G_k| = \log_p |F : R_k| + \log_p |R_k : N_k| = k + p^k + \log_p |R_k : N_k|.$$

By construction, R_k/N_k is elementary abelian of exponent p. Moreover, (3.1) shows that $\{x^{p^k}, y^p\} \cup \{[y_0, y_i] \mid 1 \leq i \leq (p^k-1)/2\}$ generates R_k modulo N_k. In order to prove that the generators are independent, we construct a factor group $\widetilde{G}_k$ of G_k that has the maximal possible logarithmic order $\log_p |\widetilde{G}_k| = p^k + k + 2 + (p^k-1)/2$.

Consider the finite p-group

$$M = \langle \widetilde{y}_0, \widetilde{y}_1, \ldots, \widetilde{y}_{p^k-1}\rangle = E/[\Phi(E), E]\Phi(E)^p,$$

where E is a free pro-p group on p^k generators with Frattini subgroup $\Phi(E) = [E, E]E^p$. Then the images of $\widetilde{y}_0, \ldots, \widetilde{y}_{p^k-1}$ generate independently the elementary abelian quotient $M/\Phi(M)$ and the commutators $[\widetilde{y}_i, \widetilde{y}_j]$, for $0 \leq i < j \leq p^k - 1$, together with the pth powers $\widetilde{y}_0^{\,p}, \ldots, \widetilde{y}_{p^k-1}^{\,p}$ generate independently the elementary abelian group $\Phi(M)$. The latter can be checked by considering homomorphisms from M onto groups of the form $C_p^{\,p^k-1} \times C_{p^2}$ and $C_p^{\,p^k-2} \times \mathrm{Heis}(\mathbb{F}_p)$, where $\mathrm{Heis}(\mathbb{F}_p)$ denotes the group of upper unitriangular 3×3 matrices over $\mathbb{F}_p$. Next consider the action of the cyclic group $X = \langle \widetilde{x} \rangle \cong C_{p^{k+1}}$, with kernel $\langle \widetilde{x}^{p^k} \rangle \cong C_p$, on M that is induced by

$$\widetilde{y}_i^{\widetilde{x}} = \begin{cases} \widetilde{y}_{i+1} & \text{if } 0 \le i \le p^k - 2, \\ \widetilde{y}_0 & \text{if } i = p^k - 1. \end{cases}$$

This induces a permutation action on our chosen basis for the elementary abelian group $\Phi(M)$; the orbits are given by

$$[\widetilde{y}_i, \widetilde{y}_j] \equiv_X [\widetilde{y}_{i'}, \widetilde{y}_{j'}] \quad \longleftrightarrow \quad j - i \equiv_{p^k} j' - i' \qquad \text{and} \qquad \widetilde{y}_0^{\,p} \equiv_X \ldots \equiv_X \widetilde{y}_{p^k-1}^{\,p}.$$

We define $\widetilde{M} = M/[\Phi(M), X]$ and, for simplicity, continue to write $\widetilde{y}_0, \ldots, \widetilde{y}_{p^k-1}$ for the images of these elements in $\widetilde{M}$. Then

- the images of $\widetilde{y}_0, \ldots, \widetilde{y}_{p^k-1}$ generate independently the elementary abelian quotient $\widetilde{M}/\Phi(\widetilde{M})$ and
- the elements $[\widetilde{y}_0, \widetilde{y}_i]$, for $1 \le i \le (p^k - 1)/2$, together with $\widetilde{y}_0^{\,p}$ generate independently the elementary abelian group $\Phi(\widetilde{M})$.

In particular, this yields $\log_p |\widetilde{M}| = p^k + (p^k - 1)/2 + 1$.

Finally, we put $\widetilde{y} = \widetilde{y}_0$ and form the semidirect product

$$\widetilde{G}_k = \langle \widetilde{x}, \widetilde{y} \rangle = X \ltimes \widetilde{M}$$

with the induced action. Upon replacing x, y by $\widetilde{x}, \widetilde{y}$, we see that all the defining relations of G_k in (3.3) are valid in $\widetilde{G}_k$. Since $\log_p |G_k| \le p^k + k + 2 + (p^k - 1)/2 = \log_p |\widetilde{G}_k|$, we conclude that $G_k \cong \widetilde{G}_k$. □

Our next aim is to prove the following structural result.

Proposition 5.2. *In the set-up from Section 3, for $k \ge 2$, the subgroup $G_k^{p^k} \le G_k$ is elementary abelian and central in G_k; it is generated independently by x^{p^k}, $w = y_{p^k-1} \cdots y_1 y_0$ and $v = w \cdot y_{p^k-1}^{-1} \cdots y_1^{-1} y_0^{-1}$.*

Consequently

$$G_k^{p^k} \cong C_p \times C_p \times C_p, \qquad \log_p |G_k : G_k^{p^k}| = \log_p |G_k| - 3$$

and

$$G_k/G_k^{p^k} \cong \langle x, y \mid x^{p^k},\ y^{p^2},\ [y^p, x],\ w(x,y),\ v(x,y);$$
$$[y_0, y_i]^p,\ [y_0, y_i, x],\ [y_0, y_i, y] \quad \text{for} 1 \le i \le \tfrac{p^k-1}{2} \rangle.$$

The proof requires a series of lemmata.

Lemma 5.3. *The elements*

$$w = y_{p^k-1} \cdots y_1 y_0 \qquad \text{and} \qquad w' = y_{p^k-1}^{-1} \cdots y_1^{-1} y_0^{-1}$$

are of order p in G_k and lie in $G_k^{p^k} \cap Z(G_k)$.

Proof. Recall that $H_k = \langle y_0, y_1, \ldots, y_{p^k-1} \rangle Z_k \le G_k$ and observe that $[H_k, H_k]$ is a central subgroup of exponent p in G_k. Furthermore, $[y^p, x] = 1$ implies $y_{p^k-1}^p = \ldots = y_0^p$ in G_k. Thus (2.1) yields

$$w^p = y_{p^k-1}^p \cdots y_1^p y_0^p = y^{p^{k+1}} = 1.$$

As $w \ne 1$ we deduce that w has order p. Likewise one shows that w' has order p.

Clearly, $w = x^{-p^k}(xy)^{p^k}$ and $w' = x^{-p^k}(xy^{-1})^{p^k}$ lie in $G_k^{p^k}$. In order to prove that w is central, it suffices to check that w commutes with the generators x and y of G_k. First we observe that, for $1 \le i \le p^k - 1$, the relation $[y_0, y_i, x] = 1$ implies

$$[y_0, y_{p^k-i}]^{-1} = [y_{p^k-i}, y_0] = [y_0, y_i]^{x^{-i}} = [y_0, y_i] \qquad \text{in } G_k. \tag{5.2}$$

Since $[H_k, H_k]$ is central in G_k, we deduce inductively that

$$\begin{aligned}
[w, x] &= (y_{p^k-1} \cdots y_1 y_0)^{-1} (y_{p^k-1} \cdots y_1 y_0)^x \\
&= y_0^{-1} y_1^{-1} \cdots y_{p^k-2}^{-1} \cdot y_{p^k-1}^{-1} y_0 y_{p^k-1} \cdot y_{p^k-2} \cdots y_2 y_1 \\
&= y_0^{-1} y_1^{-1} \cdots y_{p^k-2}^{-1} \cdot y_0 [y_0, y_{p^k-1}] \cdot y_{p^k-2} \cdots y_2 y_1 \\
&= y_0^{-1} y_1^{-1} \cdots y_{p^k-3}^{-1} \cdot y_{p^k-2}^{-1} y_0 y_{p^k-2} \cdot y_{p^k-3} \cdots y_2 y_1 \cdot [y_0, y_{p^k-1}] \\
&\ \ \vdots \\
&= [y_0, y_1][y_0, y_2] \cdots [y_0, y_{p^k-2}][y_0, y_{p^k-1}] \\
&= 1 \qquad\qquad \text{by (5.2)}.
\end{aligned}$$

Likewise, using the relation $[y_0, y_i, y] = 1$ and (5.2), we obtain

$$[w, y] = [y_{p^k-1} \cdots y_1 y_0, y_0] = [y_{p^k-1}, y_0][y_{p^k-2}, y_0] \cdots [y_1, y_0] = 1.$$

A similar computation can be carried out for w'. ☐

Lemma 5.4. *Putting*

$$v = ww' = y_{p^k-1} \dots y_1 y_0 \cdot y_{p^k-1}^{-1} \dots y_1^{-1} y_0^{-1},$$

the subgroup $\langle x^{p^k}, w, v \rangle \le G_k$ is isomorphic to $C_p \times C_p \times C_p$ and lies in $G_k^{p^k} \cap Z(G_k)$.

Proof. From the presentation (3.3) and from Lemma 5.3 it is clear that the subgroup $\langle x^{p^k}, w, v \rangle \le G_k$ is elementary abelian and lies in $G_k^{p^k} \cap Z(G_k)$. Furthermore, in order to prove that $\langle x^{p^k}, w, v \rangle \cong C_p \times C_p \times C_p$, it suffices to establish that $v \ne 1$.

Upon a similar rearrangement and cancellation as in the proof of Lemma 5.3, we obtain

$$v = \prod_{i=0}^{p^k-2} [y_i, y_{p^k-1}^{-1}][y_i, y_{p^k-2}^{-1}] \cdots [y_i, y_{i+1}^{-1}].$$

Recall that all commutators appearing in the above product are central in G_k. In particular, we have $[y_0, y_{p^k-j}] = [y_0, y_{p^k-j}]^{x^i} = [y_i, y_{p^k-j+i}]$, for $1 \le j \le p^k - 1$ and $1 \le i \le j - 1$. This gives

$$\begin{aligned}
v &= [y_0, y_{p^k-1}^{-1}]\,[y_0, y_{p^k-2}^{-1}]^2 \cdots [y_0, y_1^{-1}]^{p^k-1} \\
&= [y_0, y_{p^k-1}]^{-1}\,[y_0, y_{p^k-2}]^{-2} \cdots [y_0, y_1]^{1-p^k} \\
&= [y_0, y_1]\,[y_0, y_2]^2 \cdots [y_0, y_{(p^k-1)/2}]^{\frac{p^k-1}{2}} \\
&\qquad \cdot [y_0, y_{(p^k-1)/2}]^{(p^k-1)/2} \cdots [y_0, y_2]^2\,[y_0, y_1] \qquad\qquad \text{by (5.2)} \\
&= [y_0, y_1]^2\,[y_0, y_2]^4 \cdots [y_0, y_{(p^k-1)/2}]^{p^k-1}.
\end{aligned}$$

Taking note of the second statement in Lemma 5.1, it follows that $v \ne 1$. ☐

Lemma 5.5. *The group $\gamma_2(G_k) \le G_k$ has exponent p.*

Proof. Recall that $H_k = \langle y_0, y_1, \ldots, y_{p^k-1}\rangle Z_k \le G_k$ satisfies: $[H_k, H_k]$ is a central subgroup of exponent p in G_k. Since p is odd, (2.1) shows that it suffices to prove that $[y, x]$ has order p. But $[y, x] = y_0^{-1} y_1$; thus (2.1) and $y_0^p = x^{-1} y_0^p x = y_1^p$ imply $[y, x]^p = y_0^{-p} y_1^p = 1$. □

Lemma 5.6. *The group G_k has nilpotency class p^k, and $\gamma_m(G_k)/\gamma_{m+1}(G_k)$ is elementary abelian of rank at most 2 for $2 \le m \le p^k$.*

Proof. Let $2 \le m \le p^k$. Since G_k is a central extension of Z_k by W_k, we deduce from Proposition 2.6 that

$$\gamma_m(G_k) = \langle [y, x, \overset{m-1}{\ldots}, x], [y, x, \overset{m-2}{\ldots}, x, y] \rangle \; \gamma_{m+1}(G_k),$$

and Lemma 5.5 shows that $\gamma_m(G_k)/\gamma_{m+1}(G_k)$ is elementary abelian of rank at most 2. Again by Proposition 2.6, the nilpotency class of G_k is at least p^k. Moreover, $\gamma_{p^k}(G_k) Z_k = \langle w \rangle Z_k$, where $w \in Z(G_k)$ by Lemma 5.3. We conclude that G_k has nilpotency class precisely p^k. □

Lemma 5.7. *The group G_k satisfies*

$$G_k^p \subseteq \langle x^p, y^p \rangle \gamma_p(G_k) \qquad \textit{and} \qquad G_k^{p^j} \subseteq \langle x^{p^j} \rangle \gamma_{p^j}(G_k) \quad \textit{for } j \ge 2.$$

Proof. Recall that $H_k = \langle y_0, y_1, \ldots, y_{p^k-1}\rangle Z_k \le G_k$ has exponent p^2, and observe that Proposition 2.5 together with Lemma 5.5 yields $H_k^p = \langle y^p \rangle$. Every element $g \in G$ is of the form $g = x^m h$, with $0 \le m < p^{k+1}$ and $h \in H_k$. Using (2.3), based on Proposition 2.5 and Lemma 5.5, we conclude that

$$g^p = (x^m h)^p \equiv x^{mp} h^p \in \langle x^p, y^p \rangle \mod \gamma_p(G_k),$$

and for $j \ge 2$,

$$g^{p^j} = (x^m h)^{p^j} \equiv x^{mp^j} h^{p^j} = x^{mp^j} \in \langle x^{p^j} \rangle \mod \gamma_{p^j}(G_k). \qquad □$$

Proof of Proposition 5.2. Apply Lemmata 5.4, 5.6 and 5.7. □

From Lemma 5.1 and Proposition 5.2 we deduce that

$$\log_p |G : G^{p^k}| = \log_p |G_k : G_k^{p^k}| = \tfrac{1}{2}(3p^k + 2k - 3).$$

On the other hand, we observe from Proposition 2.6 that

$$\log_p |G : ZG^{p^k}| = \log_p |W_k : W_k^{p^k}| = p^k + k - 1,$$

hence

$$\log_p |ZG^{p^k} : G^{p^k}| = \tfrac{1}{2}(3p^k + 2k - 3) - (p^k + k - 1) = \tfrac{1}{2}(p^k - 1).$$

Thus (5.1) follows from

$$\varliminf_{i \to \infty} \frac{\log_p |ZG^{p^i} : G^{p^i}|}{\log_p |G : G^{p^i}|} = \lim_{i \to \infty} \frac{\frac{1}{2}(p^i - 1)}{\frac{1}{2}(3p^i + 2i - 3)} = 1/3. \tag{5.3}$$

Remark 5.8. In the literature, one sometimes encounters a variant of the p-power series, the *iterated* p-power series of G which is recursively given by

$$\mathcal{I} \colon I_0(G) = G, \quad \text{and} \quad I_j(G) = I_{j-1}(G)^p \quad \text{for } j \ge 1.$$

By a small modification of the proof of Lemma 5.7 we obtain inductively

$$I_j(G_k) \subseteq \big(\langle x^{p^{j-1}} \rangle \gamma_{p^{j-1}}(G_k) \big)^p \subseteq \langle x^{p^j} \rangle \gamma_{p^j}(G_k) \quad \text{for } j \ge 2,$$

based on the commutator identities (2.3) for $r = 1$. With Proposition 5.2 and Lemma 5.6 this yields $G_k^{p^k} \subseteq I_k(G_k) \subseteq \langle x^{p^k}\rangle\gamma_{p^k}(G_k) = G_k^{p^k}$. We conclude that the p-power series $\mathcal{P}$ and the iterated p-power series $\mathcal{I}$ of G coincide.

One may further note another natural filtration series $\mathcal{N}$: N_i, $i \in \mathbb{N}_0$, of G, consisting of the open normal subgroups defined in Section 3, where we set $N_0 = G$. As $N_i \le G^{p^i}$ with $\log_p|G^{p^i} : N_i| \le 4$ for all $i \in \mathbb{N}_0$, we see that the filtration series $\mathcal{P}$ and $\mathcal{N}$ induce the same Hausdorff dimension function on G.

6. The normal Hausdorff spectra of G with respect to the lower p-series, the dimension subgroup series and the Frattini series

We continue to use the notation set up in Section 3 and work with the finite quotients G_k, $k \in \mathbb{N}$, of the pro-p group G. Our aim is to pin down the lower central series, the lower p-series, the dimension subgroup series and the Frattini series of G_k. Subsequently, it will be easy to complete the proof of Theorem 1.1.

Proposition 6.1. *The group G_k is nilpotent of class p^k; its lower central series satisfies*

$$G_k = \gamma_1(G_k) = \langle x, y\rangle\, \gamma_2(G_k) \quad \textit{with} \quad G_k/\gamma_2(G_k) \cong C_{p^{k+1}} \times C_{p^2}$$

and, for $1 \le i \le (p^k-1)/2$,

$$\gamma_{2i}(G_k) = \langle [y, x, \overset{2i-1}{\ldots}, x]\rangle\, \gamma_{2i+1}(G_k),$$
$$\gamma_{2i+1}(G_k) = \langle [y, x, \overset{2i}{\ldots}, x], [y, x, \overset{2i-1}{\ldots}, x, y]\rangle\, \gamma_{2i+2}(G_k)$$

with

$$\gamma_{2i}(G_k)/\gamma_{2i+1}(G_k) \cong C_p \quad \textit{and} \quad \gamma_{2i+1}(G_k)/\gamma_{2i+2}(G_k) \cong C_p \times C_p.$$

Proof. By Lemma 5.6 the nilpotency class of G_k is p^k. From $G_k = \langle x, y\rangle$ it is clear that $\gamma_2(G_k) = \langle [x, y]\rangle\gamma_3(G_k)$, and (3.3) gives $G_k/\gamma_2(G_k) \cong C_{p^{k+1}} \times C_{p^2}$. From Lemma 5.1 we know that

$$\log_p|G_k| = (3p^k + 2k + 3)/2 = ((k+1) + 2) + \tfrac{p^k-1}{2}(1+2),$$

and the proof of Lemma 5.6 shows that

$$\gamma_m(G_k) = \langle [y, x, \overset{m-1}{\ldots}, x], [y, x, \overset{m-2}{\ldots}, x, y]\rangle\, \gamma_{m+1}(G_k) \qquad \text{for } 2 \le m \le p^k.$$

Consequently, it suffices to prove that $[y, x, \overset{m-2}{\ldots}, x, y] \in \gamma_{m+1}(G_k)$ whenever m is even. More generally, we consider the elements

$$b_{j,m} = [[y, x, \overset{m-2}{\ldots}, x]^{x^j}, y] \qquad \text{for } 2 \le m \le p^k \text{ and } j \in \mathbb{N}_0.$$

Writing $e_i = [y_0, y_i] \in Z_k \subseteq Z(G_k)$ for $i \in \mathbb{Z}$, we recall from Lemma 5.1 that

$$b_{j,m} \in [H_k, H_k] = \langle e_i \mid 1 \le i \le \tfrac{p^k-1}{2}\rangle \cong C_p \times \overset{\frac{p^k-1}{2}}{\ldots} \times C_p.$$

Induction on m shows that

$$[y, x, \overset{m-2}{\ldots}, x] \equiv \prod_{i=0}^{m-2} y_i^{(-1)^{m+i}\binom{m-2}{i}} \quad \text{modulo } Z_k \subseteq Z(G_k),$$

and we deduce that

$$b_{j,m} = \Big[\prod\nolimits_{i=0}^{m-2} y_{j+i}^{(-1)^{m+i}\binom{m-2}{i}}, y\Big] = \prod\nolimits_{i=0}^{m-2} e_{j+i}^{(-1)^{m+i+1}\binom{m-2}{i}}. \tag{6.1}$$

The identities

$$\binom{m-2}{i} - 2\binom{m-1}{i} + \binom{m}{i} = \binom{m-2}{i-2}$$

imply that

$$b_{j,m} \equiv b_{j,m} b_{j,m+1}^2 b_{j,m+2} = b_{j+2,m} \quad \text{modulo } \gamma_{m+1}(G_k). \tag{6.2}$$

Now suppose that m is even, and recall that $p \neq 2$. From (6.2) we obtain inductively $[y, x, \overset{m-2}{\ldots}, x, y] = b_{0,m} \equiv b_{j_0,m}$ modulo $\gamma_{m+1}(G_k)$ for

$$j_0 = \begin{cases} \frac{p^k+1}{2} - \frac{m}{2} & \text{if } p^k + 1 - m \equiv_4 0, \\ \frac{p^k+3}{2} - \frac{m}{2} & \text{if } p^k + 1 - m \equiv_4 2. \end{cases}$$

Consequently, it suffices to prove that $b_{j_0,m} \in \gamma_{m+1}(G_k)$. First suppose that $p^k + 1 \equiv_4 m$ and hence $j_0 = \frac{p^k+1}{2} - \frac{m}{2}$. From (6.1) and (5.2) we see that

$$\begin{aligned} b_{j_0,m} &= \prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-2}{i}} \prod\nolimits_{i=m/2}^{m-2} e_{p^k-(j_0+i)}^{(-1)^{i}\binom{m-2}{i}} \\ &= \prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-2}{i}} \prod\nolimits_{i=m/2}^{m-2} e_{j_0+(m-1-i)}^{(-1)^{m-i}\binom{m-2}{(m-1-i)-1}} \\ &= \prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-2}{i}} \prod\nolimits_{i'=1}^{m/2-1} e_{j_0+i'}^{(-1)^{i'+1}\binom{m-2}{i'-1}} \\ &= \prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-1}{i}} \end{aligned}$$

and similarly

$$\begin{aligned} b_{j_0,m+1}^{-1} &= \prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-1}{i}} \prod\nolimits_{i=m/2}^{m-1} e_{p^k-(j_0+i)}^{(-1)^{i}\binom{m-1}{i}} \\ &= \prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-1}{i}} \prod\nolimits_{i=m/2}^{m-1} e_{j_0+(m-1-i)}^{(-1)^{m-i}\binom{m-1}{m-1-i}} \\ &= \prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-1}{i}} \prod\nolimits_{i'=0}^{m/2-1} e_{j_0+i'}^{(-1)^{i'+1}\binom{m-1}{i'}} \\ &= \Big(\prod\nolimits_{i=0}^{m/2-1} e_{j_0+i}^{(-1)^{i+1}\binom{m-1}{i}}\Big)^2 \end{aligned}$$

Hence $b_{j_0,m}^2 = b_{j_0,m+1}^{-1} \in \gamma_{m+1}(G_k)$, and $p \neq 2$ implies $b_{j_0,m} \in \gamma_{m+1}(G_k)$.

In the remaining case $p^k+1 \equiv_4 m+2$ we have $j_0 = \frac{p^k+3}{2} - \frac{m}{2}$, and a slight variation of the argument above shows that $b_{j_0,m}^2 = b_{j_0-1,m+1}$, hence $b_{j_0,m} \in \gamma_{m+1}(G_k)$. □

Corollary 6.2. *For $2 \leq m \leq p^k$ and $\nu(m) = \lfloor \frac{1}{2}(p^k - m + 2) \rfloor$, we have*

$$\gamma_m(G_k) \cap Z_k = \langle [y, x, \overset{2j-1}{\ldots}, x, y] \mid \lfloor m/2 \rfloor \leq j \leq (p^k-1)/2 \rangle \cong C_p^{\nu(m)}$$

and $\gamma_m(G_k) \cap Z(G_k) = \langle [y, x, \overset{p^k-1}{\ldots}, x] \rangle \times (\gamma_m(G_k) \cap Z_k) \cong C_p^{\nu(m)+1}$. In particular,

$$[y, x, \overset{m-2}{\ldots}, x, y] \in \langle [y, x, \overset{2j-1}{\ldots}, x, y] \mid m/2 \leq j \leq (p^k-1)/2 \rangle \qquad \text{for } m \equiv_2 0.$$

Proof. Clearly, all non-trivial elements of the form $[y, x, \ldots, x, y]$ are central and of order p. By Proposition 6.1 and Lemma 5.5, also $[y, x, \overset{p^k-1}{\ldots}, x]$ is central and of order p. Moreover, Proposition 6.1 shows that every $g \in \gamma_2(G_k)$ can be written as

$$g = \prod\nolimits_{i=1}^{p^k-1} [y, x, \overset{i}{\ldots}, x]^{\alpha(i)} \prod\nolimits_{j=1}^{(p^k-1)/2} [y, x, \overset{2j-1}{\ldots}, x, y]^{\beta(j)},$$

where $\alpha(i), \beta(j) \in \{0, 1, \ldots, p-1\}$ are uniquely determined by g. Furthermore, g is central if and only if $\alpha(i) = 0$ for $1 \le i \le p^k - 2$, and $g \in Z_k$ if and only if $\alpha(i) = 0$ for $1 \le i \le p^k - 1$. $\square$

Corollary 6.3. *The lower p-series of G_k has length p^k and satisfies:*

$$G_k = P_1(G_k) = \langle x, y \rangle \; P_2(G_k) \qquad \textit{with} \quad G_k/P_2(G_k) \cong C_p \times C_p,$$
$$P_2(G_k) = \langle x^p, y^p, [y,x] \rangle \; P_3(G_k) \qquad \textit{with} \quad P_2(G_k)/P_3(G_k) \cong C_p \times C_p \times C_p,$$

and, for $3 \le i \le p^k$, the ith term is $P_i(G_k) = \langle x^{p^{i-1}} \rangle \gamma_i(G_k)$ so that

$$P_i(G_k) = \begin{cases} \langle x^{p^{i-1}}, [y, x, \overset{i-1}{\ldots}, x] \rangle \; P_{i+1}(G_k) & \textit{if } i \equiv_2 0 \textit{ and } i \le k+1, \\ \langle x^{p^{i-1}}, [y, x, \overset{i-1}{\ldots}, x], [y, x, \overset{i-2}{\ldots}, x, y] \rangle \; P_{i+1}(G_k) & \textit{if } i \equiv_2 1 \textit{ and } i \le k+1, \\ \langle [y, x, \overset{i-1}{\ldots}, x] \rangle \; P_{i+1}(G_k) & \textit{if } i \equiv_2 0 \textit{ and } i > k+1, \\ \langle [y, x, \overset{i-1}{\ldots}, x], [y, x, \overset{i-2}{\ldots}, x, y] \rangle \; P_{i+1}(G_k) & \textit{if } i \equiv_2 1 \textit{ and } i > k+1 \end{cases}$$

with

$$P_i(G_k)/P_{i+1}(G_k) \cong \begin{cases} C_p \times C_p & \textit{if } i \equiv_2 0 \textit{ and } i \le k+1, \\ C_p \times C_p \times C_p & \textit{if } i \equiv_2 1 \textit{ and } i \le k+1, \\ C_p & \textit{if } i \equiv_2 0 \textit{ and } i > k+1, \\ C_p \times C_p & \textit{if } i \equiv_2 1 \textit{ and } i > k+1. \end{cases}$$

Proof. The descriptions of $G_k/P_2(G_k)$ and $P_2(G_k)/P_3(G_k)$ are straightforward. Let $i \ge 3$. Clearly, $P_i(G_k) \supseteq \langle x^{p^{i-1}} \rangle \gamma_i(G_k)$. In view of Proposition 6.1, it suffices to prove that $x^{p^{i-1}}$ is central modulo $\gamma_{i+1}(G_k)$. Indeed, from Lemma 5.5 and Proposition 2.5 (recall that $p > 2$) we obtain

$$[x^{p^{i-1}}, y] \equiv [x, y]^{p^{i-1}} = 1 \quad \text{modulo } \gamma_{p^{i-1}+1}(G_k) \subseteq \gamma_{i+1}(G_k). \qquad \square$$

Corollary 6.4. *The dimension subgroup series of G_k has length p^k. For $1 \le i \le p^k$, the ith term is $D_i(G_k) = G_k^{p^{l(i)}} \gamma_i(G_k)$, where $l(i) = \lceil \log_p i \rceil$.*

Furthermore, if i is not a power of p, equivalently if $l(i+1) = l(i)$, then $D_i(G_k)/D_{i+1}(G_k) \cong \gamma_i(G_k)/\gamma_{i+1}(G_k)$ so that

$$D_i(G_k) = \begin{cases} \langle [y, x, \overset{i-1}{\ldots}, x] \rangle D_{i+1}(G_k) & \textit{if } i \equiv_2 0, \\ \langle [y, x, \overset{i-1}{\ldots}, x], [y, x, \overset{i-2}{\ldots}, x, y] \rangle D_{i+1}(G_k) & \textit{if } i \equiv_2 1, \end{cases}$$

with

$$D_i(G_k)/D_{i+1}(G_k) \cong \begin{cases} C_p & \textit{if } i \equiv_2 0, \\ C_p \times C_p & \textit{if } i \equiv_2 1 \end{cases}$$

whereas if $i = p^l$ is a power of p, equivalently if $l(i+1) = l+1$ for $l = l(i)$, then $D_i(G_k)/D_{i+1}(G_k) \cong \langle x^{p^l} \rangle / \langle x^{p^{l+1}} \rangle \times \langle y^{p^l} \rangle / \langle y^{p^{l+1}} \rangle \times \gamma_i(G_k)/\gamma_{i+1}(G_k)$ so that

$$D_1(G_k) = \langle x, y \rangle D_2(G_k),$$
$$D_p(G_k) = \langle x^p, y^p, [y, x, \overset{p-1}{\ldots}, x], [y, x, \overset{p-2}{\ldots}, x, y] \rangle D_{p+1}(G_k),$$
$$D_i(G_k) = \langle x^{p^l}, [y, x, \overset{i-1}{\ldots}, x], [y, x, \overset{i-2}{\ldots}, x, y] \rangle D_{i+1}(G_k)$$

with

$$D_i(G_k)/D_{i+1}(G_k) \cong \begin{cases} C_p \times C_p & \textit{if } i = 1, \textit{ equivalently if } l = 0, \\ C_p \times C_p \times C_p \times C_p & \textit{if } i = p, \textit{ equivalently if } l = 1, \\ C_p \times C_p \times C_p & \textit{if } i = p^l \textit{ with } 2 \le l \le k. \end{cases}$$

In particular, for $p^{k-1}+1 \le i \le p^k$ and thus $l(i) = k$,

$$D_i(G_k) = G_k^{p^k} \gamma_i(G_k) = \langle x^{p^k} \rangle \, \gamma_i(G_k),$$

so that

$$\log_p |D_i(G_k)| = \log_p |\gamma_i(G_k)| + 1.$$

Proof. For $i \in \mathbb{N}$ write $l(i) = \lceil \log_p i \rceil$. From [2, Thm. 11.2] and Lemma 5.5 we obtain $D_i(G_k) = G_k^{p^{l(i)}} \gamma_i(G_k)$. In particular, $D_i(G_k) = 1$ for $i > p^k$, by Proposition 6.1 and Corollary 6.3.

Now suppose that $1 \le i \le p^k$ and put $l = l(i)$. From Lemma 5.7 we observe that $G_k^{p^l} \cap \gamma_i(G_k) \subseteq \gamma_{p^l}(G_k)$. If $l(i+1) = l$ then $\gamma_{p^l}(G_k) \subseteq \gamma_{i+1}(G_k)$, and hence

$$\begin{aligned} D_i(G_k)/D_{i+1}(G_k) &= G_k^{p^l} \gamma_i(G_k) / G_k^{p^l} \gamma_{i+1}(G_k) \\ &\cong \gamma_i(G_k) / (G_k^{p^l} \cap \gamma_i(G_k)) \gamma_{i+1}(G_k) \\ &\cong \gamma_i(G_l)/\gamma_{i+1}(G_l). \end{aligned}$$

Now suppose that $l(i+1) = l+1$, equivalently $i = p^l$. We observe that, modulo H_k, the ith factor of the dimension subgroup series is

$$D_i(G_k)H_k/D_{i+1}(G_k)H_k = \langle x^{p^l} \rangle H_k / \langle x^{p^{l+1}} \rangle H_k \cong C_p.$$

Comparing with the overall order of G_k, conveniently implicit in Corollary 6.3, we deduce that

$$\begin{aligned} D_i(G_k)/D_{i+1}(G_k) &= G_k^{p^l} \gamma_i(G_k) / G_k^{p^{l+1}} \gamma_{i+1}(G_k) \\ &= \langle x^{p^l}, y^{p^l} \rangle \gamma_i(G_l) / \langle x^{p^{l+1}}, y^{p^{l+1}} \rangle \gamma_{i+1}(G_l) \\ &\cong \langle x^{p^l} \rangle / \langle x^{p^{l+1}} \rangle \times \langle y^{p^l} \rangle / \langle y^{p^{l+1}} \rangle \times \gamma_i(G_l)/\gamma_{i+1}(G_l). \end{aligned}$$

All remaining assertions follow readily from Proposition 6.1. ∎

Proposition 6.5. *The Frattini series of G_k has length $k+2$ and satisfies:*

$$G_k = \Phi_0(G_k) = \langle x, y \rangle \Phi_1(G_k) \quad \textit{with} \quad G_k/\Phi_1(G_k) \cong C_p \times C_p,$$

$$\Phi_1(G_k) = \langle x^p, y^p, [y,x], [y,x,x], \ldots, [y,x,\overset{p}{\ldots},x], [y,x,y] \rangle \Phi_2(G_k)$$
$$\textit{with} \quad \Phi_1(G_k)/\Phi_2(G_k) \cong C_p^{p+3},$$

and, for $2 \le i \le k$, the ith term is

$$\begin{aligned} \Phi_i(G_k) = \langle x^{p^i}, &[y,x,\overset{\nu(i)}{\ldots},x], [y,x,\overset{\nu(i)+1}{\ldots},x], \ldots, [y,x,\overset{\nu(i+1)-1}{\ldots},x], \\ &[y,x,\overset{2\nu(i-1)+1}{\ldots},x,y], [y,x,\overset{2\nu(i-1)+3}{\ldots},x,y], \ldots, [y,x,\overset{2\nu(i)-1}{\ldots},x,y] \rangle \Phi_{i+1}(G_k) \end{aligned}$$

$$\textit{with} \quad \Phi_i(G_k)/\Phi_{i+1}(G_k) \cong \begin{cases} C_p^{p^i+p^{i-1}+1} & \textit{for } i \ne k, \\ C_p^{p^k+1-(p^{k-1}-1)/(p-1)} & \textit{for } i = k, \end{cases}$$

where

$$\nu(j) = \min\left\{ {}^{(p^j-1)}/{}_{(p-1)}, p^k \right\} = \begin{cases} {}^{(p^j-1)}/{}_{(p-1)} & \textit{for } 1 \le j \le k, \\ p^k & \textit{for } j = k+1; \end{cases}$$

lastly,

$$\Phi_{k+1}(G_k) = \langle [y,x,\overset{2\nu(k)+1}{\ldots},x,y], [y,x,\overset{2\nu(k)+3}{\ldots},x,y], \ldots, [y,x,\overset{p^k-2}{\ldots},x,y] \rangle$$
$$\textit{with} \quad \Phi_{k+1}(G_k) \cong C_p^{(p^{k+1}-3p^k-p+3)/(2(p-1))}.$$

Proof. For ease of notation we set $c_1 = y$ and, for $i \geq 2$,

$$c_i = [y, x, \overset{i-1}{\ldots}, x] \qquad \text{and} \qquad z_i = [c_{i-1}, y] = [y, x, \overset{i-2}{\ldots}, x, y].$$

From Lemma 5.5 we observe that $c_i^p = z_i^p = 1$ for $i \geq 2$; furthermore, the elements $z_i \in [H_k, H_k] \subseteq Z_k$ are central in G_k. We claim that

$$[c_i, c_j] \equiv z_{i+j}^{(-1)^{j-1}} \bmod \gamma_{i+j+1}(G_k) \qquad \text{for } i > j \geq 1. \tag{6.3}$$

Indeed, $[c_i, c_1] = [c_i, y] = z_{i+1}$, and, modulo $\gamma_{i+j+1}(G_k)$, the Hall–Witt identity gives

$$1 \equiv [c_i, c_{j-1}, x][c_{j-1}, x, c_i][x, c_i, c_{j-1}] \equiv [c_j, c_i][c_{i+1}, c_{j-1}]^{-1},$$

hence $[c_i, c_j] \equiv [c_{i+1}, c_{j-1}]^{-1}$ from which the result follows by induction.

We use the generators specified in the statement of the proposition to define an ascending chain $1 = L_{k+2} \leq L_{k+1} \leq \ldots \leq L_1 \leq L_0 = G_k$ so that each L_i is the desired candidate for $\Phi_i(G_k)$. For $1 \leq i \leq k+1$ we deduce from Proposition 6.1 and Corollary 6.2 that

$$L_i = \langle x^{p^i} \rangle M_i \quad \text{with} \quad M_i = \langle c_{\nu(i)+1} \rangle \gamma_{\nu(i)+2}(G_k) C_i \trianglelefteq G_k,$$

where $C_i = \langle y^{p^i} \rangle \times \langle z_j \mid 2\nu(i-1) + 3 \leq j \leq p^k \text{ and } j \equiv_2 1 \rangle$ is central in G_k. (Note that the factor $\langle y^{p^i} \rangle$ vanishes if $i \geq 2$.) Applying (2.3), based on Proposition 2.5 and Lemma 5.5, we see that $[x^{p^i}, G_k] = [x^{p^i}, H_k] \subseteq \gamma_{p^i+1}(G_k)$, hence $L_i \trianglelefteq G_k$ for $1 \leq i \leq k+1$. Using also (6.3), we see that the factor groups L_i/L_{i+1} are elementary abelian for $0 \leq i \leq k+1$. In particular, this shows that $\Phi_i(G_k) \subseteq L_i$ for $1 \leq i \leq k+2$.

Clearly, for each $i \in \{0, \ldots, k+1\}$, the value of $\log_p|L_i/L_{i+1}| = d(L_i/L_{i+1})$ is bounded by the number of explicit generators used to define L_i modulo L_{i+1}; these numbers are specified in the statement of the proposition and a routine summation shows that they add up to the logarithmic order $\log_p|G_k|$, as given in Lemma 5.1. Therefore each L_i/L_{i+1} has the expected rank and it suffices to show that $\Phi_i(G_k) \supseteq L_i$ for $1 \leq i \leq k+1$.

Let $i \in \{1, \ldots, k+1\}$. It is enough to show that the following elements which generate L_i as a normal subgroup belong to $\Phi_i(G_k)$:

$$x^{p^i}, \quad c_{\nu(i)+1}, \qquad \text{and} \quad z_j \quad \text{for} \quad 2\nu(i-1) + 3 \leq j \leq p^k \text{ with } j \equiv_2 1.$$

Clearly, $x^{p^i} \in \Phi_i(G_k)$ and, applying (2.3), based on Proposition 2.5 and Lemma 5.5, we see by induction on i that

$$c_{\nu(i)+1} = [y, x, \overset{\nu(i)}{\ldots}, x] \equiv_{\Phi_i(G_k)} [y, x, x^p, \ldots, x^{p^{i-1}}] \equiv_{\Phi_i(G_k)} 1.$$

Now let $2\nu(i-1)+3 \leq j \leq p^k$ with $j \equiv_2 1$. By Corollary 6.2 and reverse induction on j it suffices to show that z_j is contained in $\Phi_i(G_k)$ modulo $\gamma_{j+1}(G_k)$. This follows from (6.3) and the fact that $c_{\nu(i-1)+1}, c_{j-\nu(i-1)-1} \in \Phi_{i-1}(G_k)$ by induction on i. $\square$

Using Corollary 4.2, we can now complete the proof of Theorem 1.1: it suffices to compute $\mathrm{hdim}_G^{\mathcal{S}}(Z)$ and $\mathrm{hdim}_G^{\mathcal{S}}(H)$ for the standard filtration series $\mathcal{S} \in \{\mathcal{L}, \mathcal{D}, \mathcal{F}\}$.

Corollary 6.3 implies

$$\mathrm{hdim}_G^{\mathcal{L}}(Z) = \varliminf_{i\to\infty} \frac{\log_p|ZP_i(G) : P_i(G)|}{\log_p|G : P_i(G)|} = \lim_{i\to\infty} \frac{i/2}{5i/2} = 1/5, \tag{6.4}$$

$$\mathrm{hdim}_G^{\mathcal{L}}(H) = \varliminf_{i\to\infty} \frac{\log_p|HP_i(G) : P_i(G)|}{\log_p|G : P_i(G)|} = \lim_{i\to\infty} \frac{3i/2}{5i/2} = 3/5. \tag{6.5}$$

Corollary 6.4 implies

$$\operatorname{hdim}_G^{\mathcal{D}}(Z) = \varliminf_{i\to\infty} \frac{\log_p|ZD_i(G) : D_i(G)|}{\log_p|G : D_i(G)|} = \lim_{i\to\infty} \frac{i/2}{3i/2} = 1/3, \tag{6.6}$$

$$\operatorname{hdim}_G^{\mathcal{D}}(H) = \varliminf_{i\to\infty} \frac{\log_p|HD_i(G) : D_i(G)|}{\log_p|G : D_i(G)|} = \lim_{i\to\infty} \frac{3i/2}{3i/2} = 1.$$

Lastly, Proposition 6.5 implies

$$\operatorname{hdim}_G^{\mathcal{F}}(Z) = \varliminf_{i\to\infty} \frac{\log_p|Z\Phi_i(G) : \Phi_i(G)|}{\log_p|G : \Phi_i(G)|} = \lim_{i\to\infty} \frac{\sum_{j=1}^{i-1} p^{j-1}}{\sum_{j=1}^{i-1}(p^j + p^{j-1} + 1)} = 1/p+1, \tag{6.7}$$

$$\operatorname{hdim}_G^{\mathcal{F}}(H) = \varliminf_{i\to\infty} \frac{\log_p|H\Phi_i(G) : \Phi_i(G)|}{\log_p|G : \Phi_i(G)|} = \lim_{i\to\infty} \frac{\sum_{j=1}^{i-1}(p^j + p^{j-1})}{\sum_{j=1}^{i-1}(p^j + p^{j-1} + 1)} = 1.$$

Remark 6.6. From (5.3), (6.4), (6.5), (6.6), (6.7) and the fact that subgroups of Hausdorff dimension 1 automatically have strong Hausdorff dimension we conclude that Z and H have strong Hausdorff dimension in G with respect to all standard filtration series $\mathcal{P}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{L}$.

7. The entire Hausdorff spectra of G with respect to the standard filtration series

We continue to use the notation set up in Section 3 to study and determine the entire Hausdorff spectra of the pro-p group G, with respect to the standard filtration series $\mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L}$.

Proof of Theorem 1.3. As in Sections 2 and 3, we write $W = G/Z \cong C_p \,\hat{\wr}\, \mathbb{Z}_p$, and we denote by $\pi \colon G \to W$ the canonical projection with $\ker \pi = Z$.

First suppose that $\mathcal{S}$ is one of the filtration series $\mathcal{P}, \mathcal{D}, \mathcal{F}$ on G. By Remark 6.6, the group H has strong Hausdorff dimension 1 in G with respect to $\mathcal{S}$. As every finitely generated subgroup of H is finite, it follows from [4, Thm. 5.4] that $\operatorname{hspec}^{\mathcal{S}}(G) = [0, 1]$.

It remains to pin down the Hausdorff spectrum of G with respect to the lower p-series $\mathcal{L} \colon P_i(G)$, $i \in \mathbb{N}$, on G. By Remark 6.6, the normal subgroups $Z, H \trianglelefteq_{\mathrm{c}} G$ have strong Hausdorff dimensions $\operatorname{hdim}_G^{\mathcal{L}}(Z) = 1/5$ and $\operatorname{hdim}_G^{\mathcal{L}}(H) = 3/5$. From Corollary 2.4, Lemma 2.2 and Corollary 2.11 we deduce that $\operatorname{hspec}^{\mathcal{L}}(G)$ contains

$$S = [0, 3/5] \cup \{3/5 + 2m/5p^n \mid m, n \in \mathbb{N}_0 \text{ with } p^n/2 < m \le p^n\}.$$

Thus it suffices to show that

$$(3/5, 4/5) \subseteq \operatorname{hspec}^{\mathcal{L}}(G) \subseteq (3/5, 4/5) \cup S. \tag{7.1}$$

First we prove the second inclusion. Let $K \le_{\mathrm{c}} G$ be any closed subgroup with $\operatorname{hdim}_G^{\mathcal{L}}(K) > 3/5$. In particular, this implies $K \not\subseteq H$ and hence $KH \le_{\mathrm{o}} G$.

We denote by $\mathcal{L}|_H$ and $\mathcal{L}|_{H\pi}$ the filtration series induced by $\mathcal{L}$ on H, via intersection, and on $H\pi = HZ/Z$, via subsequent reduction modulo Z. We write $\mathcal{L}$ for the filtration series $\mathcal{L}|_W$ induced on $W = G/Z$, as it coincides with the lower p-series of the quotient group. Using Corollary 2.11 and Lemma 2.2, we see that $(K \cap H)\pi$ has strong Hausdorff dimension

$$\alpha = \operatorname{hdim}_{H\pi}^{\mathcal{L}|_{H\pi}}((K \cap H)\pi) = 2 \operatorname{hdim}_W^{\mathcal{L}}(K\pi) - 1 \in [0, 1]$$

in $H\pi$ with respect to $\mathcal{L}|_{H\pi}$. Applying Lemma 2.2 twice, we deduce that

$$\begin{aligned} \operatorname{hdim}_G^{\mathcal{L}}(K) &= \tfrac{2}{5} + \tfrac{3}{5}\operatorname{hdim}_H^{\mathcal{L}|_H}(K \cap H) \\ &\in \tfrac{2}{5} + \tfrac{3}{5}\Big(\tfrac{2}{3}\operatorname{hdim}_{H\pi}^{\mathcal{L}|_{H\pi}}\big((K\cap H)\pi\big) + [0, {}^1\!/\!{}_3]\Big) \\ &= \tfrac{2}{5}(1+\alpha) + [0, {}^1\!/\!{}_5]. \end{aligned} \tag{7.2}$$

For $\alpha < 1/2$ we obtain $\operatorname{hdim}_G^{\mathcal{L}}(K) < 4/5$ and there is nothing further to prove. Now suppose that $\alpha \geq 1/2$. It suffices to show that $K \cap Z \leq_{\mathrm{o}} Z$ and hence $\operatorname{hdim}_G^{\mathcal{L}}(K \cap Z) = 1/5$: with this extra information we can refine the analysis in (7.2) and use Corollary 2.11 once more to deduce that

$$\operatorname{hdim}_G^{\mathcal{L}}(K) = \tfrac{2}{5}(1+\alpha) + \tfrac{1}{5} = \tfrac{4}{5}\operatorname{hdim}_W^{\mathcal{L}}(K\pi) + \tfrac{1}{5} \in S.$$

Let us prove that $K \cap Z \leq_{\mathrm{o}} Z$. As $KH \leq_{\mathrm{o}} G$, we have $KH = \langle x^{p^n}\rangle H$, where $n = \log_p |G : KH| \in \mathbb{N}_0$. Using Lemma 2.2, we deduce from $\alpha \geq 1/2$ that

$$\operatorname{hdim}_W^{\mathcal{L}}((K \cap H)\pi) \geq 1/4 = \tfrac{1}{2}\operatorname{hdim}_W^{\mathcal{L}}(H\pi). \tag{7.3}$$

At this point it is useful to recall our analysis of $\operatorname{hspec}^{\mathcal{L}}(W)$ in the proof of Theorem 2.10 and also the computations carried out in the proof of Proposition 6.5, involving the elements $c_i = [y, x, \overset{i-1}{\ldots}, x]$ and $z_i = [c_{i-1}, y]$. In particular, for $i \in \mathbb{N}$ with $i \geq 3$ we have

$$(P_i(G) \cap H)\pi = \langle c_j \mid j \geq i\rangle\pi \quad \text{and} \quad P_i(G) \cap Z = \langle z_j \mid j \geq i \text{ and } j \equiv_2 1\rangle;$$

compare Corollary 6.3. From (7.3) and the proof of Theorem 2.10 we deduce that, subject to replacing K by a suitable open subgroup $\widetilde{K} = K \cap \langle x^{p^{\widetilde{n}}}\rangle H$ with $\widetilde{n} \geq n$ if necessary, we find $m \geq (p^n + 1)/2$ and $a_1, \ldots, a_m \in K \cap H$ so that

$$(K \cap H)M/M = \langle a_1, \ldots, a_m\rangle M/M \cong C_p^m, \qquad \text{where } M = (P_{p^n+1}(G) \cap H)Z,$$

and the numbers

$$d(j) = \max\{i \in \mathbb{N} \mid a_j \in (P_i(G) \cap H)Z\}, \quad 1 \leq j \leq m,$$

form a strictly increasing sequence $1 \leq d(1) < \ldots < d(m) < p^n$. Commuting $a_1, \ldots, a_m$ repeatedly with x^{p^n}, we see as in the proof of Theorem 2.10 that

$$\{d(1), \ldots, d(m)\} + p^n\mathbb{N}_0 \subseteq \{i \in \mathbb{N} \mid \exists g \in K \cap H : g \equiv_{P_{i+1}(G)Z} c_i\}.$$

For every $k \in \mathbb{N}$ with $k > p^n$ and $k \equiv_2 1$, the pigeonhole principle (Dirichlet's 'Schubfachprinzip') yields $i, j \in \mathbb{N}$ with $i > j \geq 1$ and $i + j = k$, and we find $g_i, g_j \in K \cap H$ with $g_i \equiv_{P_{i+1}(G)Z} c_i$ and $g_j \equiv_{P_{j+1}(G)Z} c_j$ so that (6.3) gives

$$z_k \equiv_{P_{k+1}(G)} [c_i, c_j]^{(-1)^{j-1}} \equiv_{P_{k+1}(G)} [g_i, g_j]^{(-1)^{j-1}} \in K \cap Z.$$

But this implies $K \cap Z \supseteq \langle z_j \mid j > p^n \text{ and } j \equiv_2 1\rangle = P_{p^n+1}(G) \cap Z$ and thus $K \cap Z \leq_{\mathrm{o}} Z$. This concludes the proof of the second inclusion in (7.1).

Finally we prove the first inclusion in (7.1). Let $\xi \in (2/5, 4/5)$. Choose $m, n \in \mathbb{N}$ such that $1 \leq m < p^n/2$ and

$$\tfrac{1}{5}\big(2 + {}^{(4m-1)}\!/\!{}_{p^n}\big) \leq \xi \leq \tfrac{1}{5}\big(3 + {}^{2m}\!/\!{}_{p^n}\big).$$

Consider the group $K = \langle x^{p^n}, y_0, y_1, \ldots, y_{m-1} \rangle$. Using the proof of Theorem 2.10 and Lemma 2.2, we show below that K has Hausdorff dimension

$$\begin{aligned} \operatorname{hdim}_G^{\mathcal{L}}(K) &= \tfrac{4}{5} \operatorname{hdim}_W^{\mathcal{L}}(K\pi) + \tfrac{1}{5} \operatorname{hdim}_Z^{\mathcal{L}|_Z}(K \cap Z) \\ &= \left(\tfrac{2}{5} + \tfrac{2}{5}\tfrac{m}{p^n}\right) + \tfrac{1}{5}\tfrac{2m-1}{p^n} \\ &= \tfrac{1}{5}\left(2 + {}^{(4m-1)}/_{p^n}\right). \end{aligned} \tag{7.4}$$

In a similar, but much more straightforward way, we see that ZK has strong Hausdorff dimension

$$\operatorname{hdim}_G^{\mathcal{L}}(ZK) = \left(\tfrac{2}{5} + \tfrac{2}{5}\tfrac{m}{p^n}\right) + \tfrac{1}{5} = \tfrac{1}{5}\left(3 + {}^{2m}/_{p^n}\right).$$

An application of [4, Thm. 5.4] yields $L \leq_{\mathrm{c}} G$ with $K \leq L \leq ZK$ such that $\operatorname{hdim}_G^{\mathcal{L}}(L) = \xi$.

The key to (7.4) consists in showing that

$$\varliminf_{i\to\infty} \frac{\log_p |KP_i(G) \cap Z : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} = \operatorname{hdim}_Z^{\mathcal{L}|_Z}(K \cap Z) = (2m-1)/p^n. \tag{7.5}$$

First we examine the lower limit on the left-hand side, restricting to indices of the form $i = p^k + 1$, $k \in \mathbb{N}$. Let $i = p^k + 1$, where $k \geq n$. Recall that $G_k = G/\langle x^{p^{k+1}}, [x^{p^k}, y] \rangle^G$ and consider the canonical projection $\varrho_k \colon G \to G_k$, $g \mapsto \overline{g}$. As before, we write $H_k = H\varrho_k$. Furthermore, we observe that $Z_k = \langle \overline{x}^{p^k} \rangle Z\varrho_k$ with $|Z_k : Z\varrho_k| = p$. By Corollary 6.3, we have

$$|H_k : H_k \cap \underbrace{P_i(G_k)}_{=1}| = |H_k| = |H : H \cap P_i(G)|$$

and hence

$$\frac{\log_p |KP_i(G) \cap Z : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} = \frac{\log_p |K\varrho_k \cap Z\varrho_k|}{\log_p |Z\varrho_k|}.$$

Observe that

$$K\varrho_k \cap H_k = \langle \overline{y_j} \mid 0 \leq j < p^k \text{ with } j \equiv_{p^n} 0, 1, \ldots, m-1 \rangle.$$

From Lemma 5.1 we see that $Z\varrho_k \cong C_p^{(p^k+1)/2}$ and further we deduce that

$$\begin{aligned} &K\varrho_k \cap Z\varrho_k \\ &\quad = \langle \{\overline{y}^p\} \cup \{[\overline{y_0}, \overline{y_j}] \mid 0 \leq j < p^k,\ j \equiv_{p^n} 0, \pm 1, \ldots, \pm(m-1),\ j \equiv_2 0\} \rangle \\ &\quad \cong C_p^{((2m-1)p^{k-n}+1)/2}. \end{aligned}$$

This yields

$$\begin{aligned} \varliminf_{i\to\infty} \frac{\log_p |KP_i(G) \cap Z : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} &\leq \varliminf_{k\to\infty} \frac{\log_p |K\varrho_k \cap Z\varrho_k|}{\log_p |Z\varrho_k|} \\ &= \lim_{k\to\infty} \frac{(2m-1)p^{k-n}+1}{p^k+1} = (2m-1)/p^n. \end{aligned}$$

In order to establish (7.5) it now suffices to prove that

$$\varliminf_{i\to\infty} \frac{\log_p |(K \cap Z)(P_i(G) \cap Z) : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} \geq (2m-1)/p^n. \tag{7.6}$$

Our analysis above yields

$$K \cap Z = \langle \{y^p\} \cup \{[y_0, y_j] \mid j \in \mathbb{N} \text{ with } j \equiv_{p^n} 0, \pm 1, \ldots, \pm(m-1)\} \rangle.$$

Setting

$$L = \langle y_j \mid j \in \mathbb{N}_0 \text{ with } j \equiv_{p^n} 0, \pm 1, \ldots, \pm(m-1) \rangle Z,$$

and recalling the notation $c_1 = y = y_0$, we conclude that

$$K \cap Z \supseteq \{[g, c_1] \mid g \in L\}.$$

Next we consider the set

$$D = \{j \in \mathbb{N} \mid \exists g \in L : g \equiv_{P_{j+1}(G)Z} c_j\}.$$

Each element y_j can be written (modulo Z) as a product

$$y_j \equiv_Z \prod_{k=0}^{j} c_{k+1}^{\beta(j,k)} \qquad \text{where } \beta(j,k) = \tbinom{j}{k},$$

using the elements $c_i = [y, x, \overset{i-1}{\ldots}, x]$ introduced in the proof of Proposition 6.5. In this product decomposition, the exponents should be read modulo p, and the elementary identity $(1+t)^{j+p^n} = (1+t)^j(1+t^{p^n})$ in $\mathbb{F}_p[\![t]\!]$ translates to

$$y_j^{-1} y_{j+p^n} = y^{-x^j} y^{x^{j+p^n}} \equiv_Z \prod_{k=0}^{j} c_{k+1+p^n}^{\beta(j,k)} \qquad \text{for all } j \in \mathbb{N};$$

compare (2.4). Inductively, we obtain

$$D = D_0 + p^n \mathbb{N}_0 \qquad \text{for } D_0 = D \cap \{1, \ldots, p^n\}.$$

Observe that $|D_0| = 2m-1$ and that, for each $k \in \mathbb{N}_0$, the set $(2kp^n + D_0) \cup ((2k+1)p^n + D_0)$ consists of $2m-1$ odd and $2m-1$ even numbers.

For each $j \in D$ with $j \equiv_2 0$ there exists $g_j \in L$ with $g_j \equiv_{P_{j+1}(G)Z} c_j$ and we deduce that

$$z_{j+1} = [c_j, c_1] \equiv_{P_{j+2}(G)} [g_j, c_1] \in K \cap Z.$$

For $i = 2p^n q + r \in \mathbb{N}$, where $q, r \in \mathbb{N}_0$ with $0 \le r < 2p^n$, the count

$$|\{j \in D \mid j \equiv_2 0 \text{ and } j < i-1\}| \ge q(2m-1) - 1$$

yields

$$\log_p |(K \cap Z)(P_i(G) \cap Z) : P_i(G) \cap Z| \ge q(2m-1) - 1.$$

From Corollary 6.3 we observe that, for $i \ge 3$,

$$\log_p |Z : P_i(G) \cap Z| = \lfloor i/2 \rfloor \le qp^n + p^n.$$

These estimates show that (7.6) holds. □

Appendix A. The case $p = 2$

When p is even, Theorems 1.1 and 1.3, and all the results of Sections 2 and 4, hold with corresponding proofs. The structural results of Sections 5 and 6 however are slightly different and we now sketch these differences below; for complete details, we refer the reader to the supplement [8].

Firstly, for $p = 2$,

$$\begin{aligned} G_k = F/N_k \cong \langle x, y \mid\ & x^{2^{k+1}},\ y^4,\ [x^{2^k}, y],\ [y^2, x]; \\ & [y_0, y_i]^2,\ [y_0, y_i, x],\ [y_0, y_i, y] \quad \text{for } 1 \le i \le 2^{k-1} \rangle \end{aligned} \tag{A.1}$$

for $k \in \mathbb{N}$, and

$$G \cong \langle x, y \mid y^4,\ [y^2, x];\ [y_0, y_i]^2,\ [y_0, y_i, x],\ [y_0, y_i, y] \quad \text{for } i \in \mathbb{N} \rangle \tag{A.2}$$

is a presentation of G as a pro-2 group.

Next, we have $\log_2 |G_k| = 2^k + 2^{k-1} + k + 2$ and the exponent of $\gamma_2(G_k)$ is 4. With regards to Lemma 5.3 , the elements

$$w = y_{2^k-1} \cdots y_1 y_0 \qquad \text{and} \qquad [w,x] = [w,y] = [y_0, y_{2^{k-1}}]$$

are of order 2 in G_k and lie in $G_k^{2^k}$. In particular the subgroup $\langle x^{2^k}, w, [w,x] \rangle$ is isomorphic to $C_2 \times C_2 \times C_2$ and lies in $G_k^{2^k}$. Hence, for $k \geq 2$,

$$G_k^{2^k} = \langle x^{2^k}, w, [w,x] \rangle \cong C_2 \times C_2 \times C_2, \qquad \log_2 |G_k : G_k^{2^k}| = \log_2 |G_k| - 3$$

and

$$G_k/G_k^{2^k} \cong \langle x, y \mid x^{2^k},\ y^4,\ [y^2,x],\ w(x,y),\ [y_0, y_{2^{k-1}}]; \\ [y_0,y_i]^2,\ [y_0,y_i,x],\ [y_0,y_i,y] \quad \text{for } 1 \leq i < 2^{k-1} \rangle.$$

Lemma 5.7 is slightly different; here the group G_k satisfies $G_k^2 \subseteq \langle x^2, y^2 \rangle \gamma_2(G_k)$ and

$$G_k^{2^j} \subseteq \langle x^{2^j}, [y, x, \overset{2^j-3}{\ldots}, x, y] \rangle \gamma_{2^j}(G_k) \subseteq \langle x^{2^j} \rangle \gamma_{2^j-1}(G_k) \quad \text{for } j \geq 2.$$

The proof is similar, but one needs the fact

$$[y, x, \overset{i}{\ldots}, x]^2 \in [H_k, H_k] \cap \gamma_{2i+1}(G_k), \quad \text{for } i \geq 1,$$

which is proved by induction, using

$$[[y, x, \overset{i-1}{\ldots}, x], x]^2 = [[y, x, \overset{i-1}{\ldots}, x]^x, [y, x, \overset{i-1}{\ldots}, x]^{-1}] \qquad \text{for } i \geq 2.$$

Furthermore, $[y, x, \overset{i}{\ldots}, x]^2 \equiv [y, x, \overset{2i-1}{\ldots}, x, y]$ modulo $\gamma_{2i+2}(G_k)$.

The group G_k is nilpotent of class $2^k + 1$; its lower central series satisfies

$$G_k = \gamma_1(G_k) = \langle x, y \rangle\, \gamma_2(G_k) \quad \text{with} \quad G_k/\gamma_2(G_k) \cong C_{2^{k+1}} \times C_4$$

and, for $1 \leq i \leq 2^{k-1}$,

$$\gamma_{2i}(G_k) = \langle [y, x, \overset{2i-1}{\ldots}, x] \rangle\, \gamma_{2i+1}(G_k),$$
$$\gamma_{2i+1}(G_k) = \begin{cases} \langle [y, x, \overset{2i}{\ldots}, x], [y, x, \overset{2i-1}{\ldots}, x, y] \rangle\, \gamma_{2i+2}(G_k) & \text{for } i \neq 2^{k-1} \\ \langle [y, x, \overset{2i}{\ldots}, x] \rangle\, \gamma_{2i+2}(G_k) & \text{for } i = 2^{k-1} \end{cases}$$

with

$$\gamma_{2i}(G_k)/\gamma_{2i+1}(G_k) \cong C_2 \quad \text{and} \quad \gamma_{2i+1}(G_k)/\gamma_{2i+2}(G_k) \cong \begin{cases} C_2 \times C_2 & \text{for } i \neq 2^{k-1} \\ C_2 & \text{for } i = 2^{k-1}. \end{cases}$$

The proof of the above is similar to that for the odd prime case, however here one takes

$$j_0 = \begin{cases} 2^{k-1} - \frac{m}{2} & \text{if } m \equiv_4 0, \\ 2^{k-1} + 1 - \frac{m}{2} & \text{if } m \equiv_4 2. \end{cases}$$

For the $m \equiv_4 0$ case, noting that $e_{2^{k-1}} = [w,x] \in \gamma_{2^k+1}(G_k)$, we have $b_{j_0,m} \equiv b_{j_0,m+1}$ modulo $\gamma_{m+1}(G_k)$. The $m \equiv_4 2$ case is similar.

The lower 2-series of G_k has length $2^k + 1$ and satisfies the corresponding form, based on the lower central series of G_k above.

The dimension subgroup series of G_k has length $2^k + 2$. For $1 \leq i \leq 2^k + 2$, the ith term is $D_i(G_k) = G_k^{2^{l(i)}} \gamma_{\lceil i/2 \rceil}(G_k)^2 \gamma_i(G_k)$, where $l(i) = \lceil \log_2 i \rceil$.

Furthermore, if i is not a power of 2, equivalently if $l(i+1) = l(i)$, then $D_i(G_k)/D_{i+1}(G_k) \cong \gamma_{\lceil i/2\rceil}(G_k)^2\gamma_i(G_k)/\gamma_{\lceil i+1/2\rceil}(G_k)^2\gamma_{i+1}(G_k)$ so that

$$D_i(G_k) = \begin{cases} \langle [y,x,\overset{i-1}{\ldots},x]\rangle D_{i+1}(G_k) & \text{if } i \equiv_2 1, \\ \langle [y,x,\overset{i-3}{\ldots},x,y],[y,x,\overset{i-1}{\ldots},x]\rangle D_{i+1}(G_k) & \text{if } i \equiv_2 0, \end{cases}$$

with

$$D_i(G_k)/D_{i+1}(G_k) \cong \begin{cases} C_2 & \text{if } i \equiv_2 1 \text{ and } i < 2^k, \\ C_2 \times C_2 & \text{if } i \equiv_2 0 \text{ and } i < 2^k, \\ 1 & \text{if } i = 2^k + 1, \\ C_2 & \text{if } i = 2^k + 2. \end{cases}$$

whereas if $i = 2^l$ is a power of 2, equivalently if $l(i+1) = l+1$ for $l = l(i)$, then $D_i(G_k)/D_{i+1}(G_k) \cong \langle x^{2^l}\rangle/\langle x^{2^{l+1}}\rangle \times \langle y^{2^l}\rangle/\langle y^{2^{l+1}}\rangle \times \langle [y,x,\overset{i-3}{\ldots},x,y]\rangle\gamma_i(G_k)/\gamma_{i+1}(G_k)$ so that

$$\begin{aligned} D_1(G_k) &= \langle x,y\rangle D_2(G_k), \\ D_2(G_k) &= \langle x^2,y^2,[y,x]\rangle D_3(G_k), \\ D_i(G_k) &= \langle x^{2^l},[y,x,\overset{i-3}{\ldots},x,y],[y,x,\overset{i-1}{\ldots},x]\rangle D_{i+1}(G_k) \end{aligned}$$

with

$$D_i(G_k)/D_{i+1}(G_k) \cong \begin{cases} C_2 \times C_2 & \text{if } i = 1, \text{ equivalently if } l = 0, \\ C_2 \times C_2 \times C_2 & \text{if } i = 2, \text{ equivalently if } l = 1, \\ C_2 \times C_2 \times C_2 & \text{if } i = 2^l \text{ with } 2 \le l \le k. \end{cases}$$

In particular, for $2^{k-1}+1 \le i \le 2^k$ and thus $l(i) = k$,

$$D_i(G_k) = G_k^{2^k}\gamma_i(G_k) = \langle x^{2^k},[y,x,\overset{2^k-3}{\ldots},x,y]\rangle\,\gamma_i(G_k),$$

so that

$$\log_2|D_i(G_k)| = \log_2|\gamma_i(G_k)| + 1.$$

Lastly, the Frattini series of G_k has the corresponding form, though it has length $k+1$.

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