PROFINITE GROUPS WITH AN AUTOMORPHISM WHOSE FIXED POINTS ARE RIGHT ENGEL

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Abstract. An element g of a group G is said to be right Engel if for every $x \in G$ there is a number $n = n(g,x)$ such that $[g,{}_n x] = 1$. We prove that if a profinite group G admits a coprime automorphism φ of prime order such that every fixed point of φ is a right Engel element, then G is locally nilpotent.

1. Introduction

Let G be a profinite group, and φ a (continuous) automorphism of G of finite order. We say for short that φ is a *coprime automorphism* of G if its order is coprime to the orders of elements of G (understood as Steinitz numbers), in other words, if G is an inverse limit of finite groups of order coprime to the order of φ. Coprime automorphisms of profinite groups have many properties similar to the properties of coprime automorphisms of finite groups. In particular, if φ is a coprime automorphism of G, then for any (closed) normal φ-invariant subgroup N the fixed points of the induced automorphism (which we denote by the same letter) in G/N are images of the fixed points in G, that is, $C_{G/N}(\varphi) = C_G(\varphi)N/N$. Therefore, if φ is a coprime automorphism of prime order q such that $C_G(\varphi) = 1$, Thompson's theorem [18] implies that G is pronilpotent, and Higman's theorem [7] implies that G is nilpotent of class bounded in terms of q.

In this paper we consider profinite groups admitting a coprime automorphism of prime order all of whose fixed points are right Engel elements. Recall that the n-Engel word $[y,{}_n x]$ is defined recursively by $[y,{}_0 x] = y$ and $[y,{}_{i+1} x] = [[y,{}_i x], x]$. An element g of a group G is said to be right Engel if for any $x \in G$ there is an integer $n = n(g,x)$ such that $[g,{}_n x] = 1$. If all elements of a group are right Engel (therefore also left Engel), then the group is called an Engel group. By a theorem of Wilson and Zelmanov [20] based on Zelmanov's results [21, 22, 23] on Engel Lie algebras, an Engel profinite group is locally nilpotent. Recall that a group is said to be locally nilpotent if every finite subset generates a nilpotent subgroup. In our main result the right Engel condition is imposed on the fixed points of a coprime automorphism of prime order.

Theorem 1.1. *Suppose that φ is a coprime automorphism of prime order of a profinite group G. If every element of $C_G(\varphi)$ is a right Engel element of G, then G is locally nilpotent.*

The proof of Theorem 1.1 begins with the observation that a group G satisfying the hypothesis is pronilpotent. Indeed, right Engel elements of a finite group are contained in the hypercentre by the well-known theorem of Baer [1]. Therefore every finite quotient of G by a φ-invariant open normal subgroup is nilpotent by Thompson's theorem [18], since

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φ acts fixed-point-freely on the quotient by the hypercentre. Assuming in addition that G is finitely generated, it remains to prove that all Sylow p-subgroups S_p of G are nilpotent with a uniform upper bound for the nilpotency class. This is achieved in two stages. First a bound for the nilpotency class of S_p depending on p is obtained for all p. Then a bound independent of p is obtained for all sufficiently large primes p. At both stages we apply Lie ring methods and the crucial tool is Zelmanov's theorem [21, 22, 23] on Lie algebras and some of its consequences. Other important ingredients include criteria for a pro-p group to be p-adic analytic in terms of the associated Lie algebra due to Lazard [11], and in terms of bounds for the rank due to Lubotzky and Mann [13], and a theorem of Bahturin and Zaicev [2] on Lie algebras admitting a group of automorphisms whose fixed-point subalgebra is PI.

2. Preliminaries

Lie rings and algebras. Products in Lie rings and algebras are called commutators. We use simple commutator notation for left-normed commutators $[x_1, \dots, x_k] = [...[x_1, x_2], \dots, x_k]$, and the short-hand for Engel commutators $[x,{}_n y] = [x, y, y, \dots, y]$ with y occurring n times. An element a of a Lie ring or a Lie algebra L is said to be ad-nilpotent if there exists a positive integer n such that $[x,{}_n a] = 0$ for all $x \in L$. If n is the least integer with this property, then we say that a is ad-nilpotent of index n.

The next theorem is a deep result of Zelmanov [21, 22, 23].

Theorem 2.1. *Let L be a Lie algebra over a field and suppose that L satisfies a polynomial identity. If L can be generated by a finite set X such that every commutator in elements of X is ad-nilpotent, then L is nilpotent.*

An important criterion for a Lie algebra to satisfy a polynomial identity is provided by the next theorem, which was proved by Bahturin and Zaicev for soluble group of automorphisms [2] (and later extended by Linchenko to the general case [12]). We use the centralizer notation for the fixed point subring $C_L(A)$ of a group of automorphisms A of L.

Theorem 2.2. *Let L be a Lie algebra over a field K. Assume that a finite group A acts on L by automorphisms in such a manner that $C_L(A)$ satisfies a polynomial identity. Assume further that the characteristic of K is either 0 or coprime with the order of A. Then L satisfies a polynomial identity.*

Both Theorems 2.1 and 2.2 admit respective quantitative versions (see for example [16]). For our purposes, we shall need the following result for Lie rings proved in [17, Proposition 2.6], which combines both versions. As usual, $\gamma_i(L)$ denotes the i-th term of the lower central series of L.

Proposition 2.3. *Let L be a Lie ring and A a finite group of automorphisms of L such that $C_L(A)$ satisfies a polynomial identity $f \equiv 0$. Suppose that L is generated by an A-invariant set of m elements such that every commutator in these elements is ad-nilpotent of index at most n. Then there exist positive integers e and c depending only on $|A|$, f, m, and n such that $e\gamma_c(L) = 0$.*

We also quote the following useful result proved in [10, Lemma 5] (although it was stated for Lie algebras in [10], the proof is the same for Lie rings).

Lemma 2.4. *Let L be a Lie ring, and M a subring of L generated by m elements such that all commutators in these elements are ad-nilpotent in L of index at most n. If M is nilpotent of class c, then for some number $\varepsilon = \varepsilon(m, n, c)$ bounded in terms of m, n, c we have* $[L, \underbrace{M, M, \dots, M}_{\varepsilon}] = 0$.

Associated Lie rings and algebras. We now remind the reader of one of the ways of associating a Lie ring with a group. A series of subgroups of a group G

$$G = G_1 \geqslant G_2 \geqslant \cdots \tag{2.1}$$

is called a *filtration* (or an *N-series*, or a *strongly central series*) if

$$[G_i, G_j] \leqslant G_{i+j} \qquad \text{for all} \quad i, j. \tag{2.2}$$

For any filtration (2.1) we can define an associated Lie ring $L(G)$ with additive group

$$L(G) = \bigoplus_i G_i/G_{i+1},$$

where the factors $L_i = G_i/G_{i+1}$ are additively written. The Lie product is defined on homogeneous elements $xG_{i+1} \in L_i$, $yG_{j+1} \in L_j$ via the group commutators by

$$[xG_{i+1},\, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}$$

and extended to arbitrary elements of $L(G)$ by linearity. Condition (2.2) ensures that this Lie product is well-defined, and group commutator identities imply that $L(G)$ with these operations is a Lie ring. If all factors G_i/G_{i+1} of a filtration (2.1) have prime exponent p, then $L(G)$ can be viewed as a Lie algebra over the field of p elements $\mathbb{F}_p$. If all terms of (2.1) are invariant under an automorphism φ of the group G, then φ naturally induces an automorphism of $L(G)$.

We shall normally indicate which filtration is used for constructing an associated Lie ring. One example of a filtration (2.1) is given by the lower central series, the terms of which are denoted by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$. It is worth noting that the corresponding associated Lie ring $L(G)$ is generated by the homogeneous component $L_1 = G/\gamma_2(G)$.

Another example, for a fixed prime number p, is the *Zassenhaus p-filtration* (also called the *p-dimension series*), which is defined by

$$G_i = \langle g^{p^k} \mid g \in \gamma_j(G),\ \ jp^k \geqslant i\rangle.$$

The factors of this filtration are elementary abelian p-groups, so the corresponding associated Lie ring $D_p(G)$ is a Lie algebra over $\mathbb{F}_p$. We denote by $L_p(G)$ the subalgebra generated by the first factor G/G_2. It is well known that the homogeneous components of $D_p(G)$ of degree s coincide with the homogeneous components of $L_p(G)$ for all s that are not divisible by p. In particular, $L_p(G)$ is nilpotent if and only if $D_p(G)$ is nilpotent. (Sometimes, the notation $L_p(G)$ is used for $D_p(G)$.)

A group G is said to satisfy a *coset identity* if there is a group word $w(x_1, \dots, x_m)$ and cosets $a_1H, \dots, a_mH$ of a subgroup $H \leqslant G$ of finite index such that $w(a_1h, \dots, a_mh) = 1$ for any $h \in H$. Wilson and Zelmanov [20] proved that if a group G satisfies a coset identity, then the Lie algebra $L_p(G)$ constructed with respect to the Zassenhaus p-filtration satisfies a polynomial identity. In fact, the proof of Theorem 1 in [20] can be slightly modified to become valid for any filtration (2.1) with abelian factors of prime exponent p and the corresponding associated Lie algebra.

Profinite groups. We always consider a profinite group as a topological group. A subgroup of a topological group will always mean a closed subgroup, all homomorphisms are continuous, and quotients are by closed normal subgroups. This also applies to taking commutator subgroups, normal closures, subgroups generated by subsets, etc. We say that a subgroup is generated by a subset X if it is generated by X as a topological group. Note that if φ is a continuous automorphism of a topological group G, then the fixed-point subgroup $C_G(\varphi)$ is closed.

Recall that a pronilpotent group is a pro-(finite nilpotent) group, that is, an inverse limit of finite nilpotent groups. For a prime p, a pro-p group is an inverse limit of finite p-groups. The Frattini subgroup $P'P^p$ of a pro-p group P is the product of the derived subgroup P' and the subgroup generated by all p-th powers of elements of P. A subset generates P (as a topological group) if and only if its image generates the elementary abelian quotient $P/(P'P^p)$. See, for example, [19] for these and other properties of profinite groups.

3. Local nilpotency of Sylow p-subgroups

In this section we prove the local nilpotency of a pro-p group satisfying the hypotheses of the main Theorem 1.1. We shall use without special references the fact that fixed points $C_{G/N}(\varphi)$ of an automorphism φ of finite coprime order in a quotient by a φ-invariant normal open subgroup N are covered by the fixed points in the group: $C_{G/N}(\varphi) = C_G(\varphi)N/N$.

Theorem 3.1. *Let p be a prime and suppose that a finitely generated pro-p group G admits an automorphism φ of prime order $q \neq p$. If every element of $C_G(\varphi)$ is a right Engel element of G, then G is nilpotent.*

We begin with constructing a normal subgroup with nilpotent quotient that will be the main focus of the proof. Recall that $h(q)$ is a function bounding the nilpotency class of a nilpotent group admitting a fixed-point-free automorphism of prime order q by Higman's theorem [7].

Lemma 3.2. *There is a finite set $S \subseteq C_G(\varphi)$ of fixed points of φ such that the quotient G/H by its normal closure $H = \langle S^G \rangle$ is nilpotent of class $h(q)$.*

Proof. In the nilpotent quotient $G/\gamma_{h(q)+2}(G)$ of the finitely generated group G every subgroup is finitely generated. Therefore there is a finite set S of elements of $C_G(\varphi)$ whose images cover all fixed points of φ in $G/\gamma_{h(q)+2}(G)$. Let $H = \langle S^G \rangle$ be the normal closure of S. Then the quotient of G by $H\gamma_{h(q)+2}(G)$ is nilpotent of class $h(q)$ by Higman's theorem, which means that $\gamma_{h(q)+1}(G) \leqslant H\gamma_{h(q)+2}(G)$. Since the group G/H is pronilpotent, it follows that $\gamma_{h(q)+1}(G) \leqslant H$, as required. □

We fix the notation for the subgroup $H = \langle S^G \rangle$ and the finite set $S \subseteq C_G(\varphi)$ given by Lemma 3.2. We aim at an application of Zelmanov's Theorem 2.1 to the associated Lie algebra of H, verifying the requisite conditions in a number of steps. The first step is to show that the quotient G/H' is nilpotent, which is achieved by the following lemma.

Lemma 3.3. *Suppose that L is a finitely generated pro-p group, M is an abelian normal subgroup equal to the normal closure $M = \langle T^L \rangle$ of a finite set T consisting of right Engel elements of L, and L/M is nilpotent. Then L is nilpotent.*

Proof. We proceed by induction on the nilpotency class of $L/C_L(M)$. The base of induction is the case where $L/C_L(M) = 1$, that is, $L = C_L(M)$; then L is obviously nilpotent.

Let $T = \{t_1, \ldots, t_k\}$. Let Z be the inverse image of the centre $Z(L/C_L(M))$ of $L/C_L(M)$. We claim that Z is nilpotent. For any fixed $z \in Z$ there are positive integers n_i such that $[t_i,_{n_i} z] = 1$. Set $n = \max_i n_i$; then $[t_i,_n z] = 1$ for all i. Moreover, for any $g \in L$ we have $[t_i^g,_n z] = [t_i,_n z]^g = 1$ since $[z, g] \in C_L(M)$. Since $M = \langle T^L \rangle$ is abelian, this implies that $[m,_n z] = 1$ for any finite product m of the elements t_i^g, $g \in G$. Since these finite products form a dense subset of M, we obtain

$$[m,_n z] = 1 \qquad \text{for any} \quad m \in M. \tag{3.1}$$

Since L/M is nilpotent and finitely generated, Z/M is nilpotent and finitely generated. Together with (3.1) this implies that Z is nilpotent. Indeed, let $Z = \langle M, z_1, \ldots, z_s \rangle$. Any sufficiently long simple commutator in the elements of M and $z_1, \ldots, z_s$ has an initial segment that belongs to M because Z/M is nilpotent. Since $Z/C_Z(M)$ is abelian, the remaining elements (which can all be assumed to be among the z_i) can be arbitrarily rearranged without changing the value of the commutator. If the commutator is sufficiently long, one of the z_i will appear sufficiently many times in a row making the commutator trivial by (3.1).

We now consider L/Z', denoting by the bar the corresponding images of subgroups and elements. Clearly, $\bar{L}$, $\bar{M}$, and $\bar{T}$ satisfy the hypotheses of the lemma. But now $\bar{Z} \leqslant C_{\bar{L}}(\bar{M})$, so the nilpotency class of $\bar{L}/C_{\bar{L}}(\bar{M})$ is less than that of $L/C_L(M)$ (unless $Z = L$ when the proof is complete). By the induction hypothesis, L/Z' is nilpotent. Together with the nilpotency of Z proved above, this implies that L is nilpotent by Hall's theorem [5]. □

Lemma 3.4. *The subgroup H is generated by finitely many right Engel elements.*

Proof. By Lemma 3.3 applied with $L = G/H'$, $M = H/H'$, and $T = S$, the quotient G/H' is nilpotent. Then H/H' is finitely generated as a subgroup of a finitely generated nilpotent group. The Frattini quotient $H/(H'H^p)$ is a finite elementary abelian p-group. Since H is generated by a set of right Engel elements, conjugates of elements of $C_G(\varphi)$, we can choose a finite subset of these elements whose images generate $H/(H'H^p)$. Then this finite set also generates the pro-p group H. □

Let $L_p(H)$ be the associated Lie algebra of H over $\mathbb{F}_p$ constructed with respect to the Zassenhaus p-filtration of H.

Proposition 3.5. *The Lie algebra $L_p(H)$ is nilpotent.*

Proof. This will follow from Zelmanov's Theorem 2.1 if we show that $L_p(H)$ satisfies a polynomial identity and is generated by finitely many elements such that all commutators in these elements are ad-nilpotent.

Lemma 3.6. *The Lie algebra $L_p(H)$ satisfies a polynomial identity.*

Proof. Note that H is a φ-invariant subgroup, since $H = \langle S^G \rangle$ for $S \subseteq C_G(\varphi)$. As a profinite Engel group, $C_H(\varphi) = H \cap C_G(\varphi)$ is locally nilpotent by the Wilson–Zelmanov theorem [20]. It follows that $C_H(\varphi)$ satisfies a coset identity on cosets of an open subgroup of $C_H(\varphi)$. For example, in the group $C_H(\varphi) \times C_H(\varphi)$ the subsets

$$E_i = \{(x, y) \in C_H(\varphi) \times C_H(\varphi) \mid [x,_i y] = 1\}$$

are closed in the product topology, and

$$C_H(\varphi) \times C_H(\varphi) = \bigcup_{i=1}^{\infty} E_i.$$

Hence by the Baire category theorem [8, Theorem 34], one of these subsets E_n contains an open subset of $C_H(\varphi) \times C_H(\varphi)$, which means that there are cosets x_0K_1, y_0K_2 of open subgroups $K_1, K_2 \leqslant C_H(\varphi)$ such that $[x, {}_n y] = 1$ for all $x \in x_0K_1$ and $y \in y_0K_2$, and therefore for all $x \in x_0(K_1 \cap K_2)$ and $y \in y_0(K_1 \cap K_2)$. Thus, $C_H(\varphi)$ satisfies a coset identity.

The intersections $C_i = C_H(\varphi) \cap H_i$ with the terms H_i of the Zassenhaus p-filtration for H form a filtration of $C_H(\varphi)$, since obviously, $[C_i, C_j] \leqslant C_{i+j}$. Let $\hat{L}_p(C_H(\varphi))$ be the Lie algebra constructed for $C_H(\varphi)$ with respect to the filtration $\{C_i\}$. Since φ is a coprime automorphism, the fixed-point subalgebra $C_{L_p(H)}(\varphi)$ is isomorphic to $\hat{L}_p(C_H(\varphi))$. We apply a version of the Wilson–Zelmanov result [20, Theorem 1], by which a coset identity on $C_H(\varphi)$ implies that $\hat{L}_p(C_H(\varphi))$ satisfies some polynomial identity. Indeed, the proof of Theorem 1 in [20] only uses the filtration property $[F_i, F_j] \leqslant F_{i+j}$ for showing that the homogeneous Lie polynomial constructed from a coset identity on a group F is an identity of the Lie algebra constructed with respect to the filtration $\{F_i\}$.

Thus, the fixed-point subalgebra $C_{L_p(H)}(\varphi)$ satisfies a polynomial identity. Hence the Lie algebra $L_p(H)$ also satisfies a polynomial identity by the Bahturin–Zaicev Theorem 2.2. □

Lemma 3.7. *The Lie algebra $L_p(H)$ is generated by finitely many elements such that all commutators in these elements are ad-nilpotent.*

Proof. By Lemma 3.4 the group H is generated by finitely many right Engel elements, say, $h_1, \dots, h_m$. Their images $\bar{h}_1, \dots, \bar{h}_m$ in the first factor H/H_2 of the Zassenhaus p-filtration of H generate the Lie algebra $L_p(H)$. Let $\bar{c}$ be some commutator in these generators $\bar{h}_i$, and c the same group commutator in the elements h_i. For every j, since $[h_j, {}_{k_j} c] = 1$ for some $k_j = k_j(c)$, we also have $[\bar{h}_j, {}_{k_j} \bar{c}] = 0$ in $L_p(H)$. We choose a positive integer s such that $p^s \geqslant \max\{k_1, \dots, k_m\}$. Then $[\bar{h}_j, {}_{p^s} \bar{c}] = 0$ for all j. In characteristic p this implies that

$$[\varkappa, {}_{p^s} \bar{c}] = 0 \tag{3.2}$$

for any commutator $\varkappa$ in the $\bar{h}_i$. This easily follows by induction on the weight of $\varkappa$ from the formula

$$[[u, v], {}_{p^s} w] = [[u, {}_{p^s} w], v] + [u, [v, {}_{p^s} w]]$$

that holds in any Lie algebra of characteristic p. This formula follows from the Leibnitz formula

$$[[u, v], {}_n w] = \sum_{i=0}^{n} \binom{n}{i} [[u, {}_i w],\, [v, {}_{n-i} w]]$$

(where $[a, {}_0 b] = a$), because the binomial coefficient $\binom{p^s}{i}$ is divisible by p unless $i = 0$ or $i = p^s$.

Since any element of $L_p(H)$ is a linear combination of commutators in the $\bar{h}_i$, equation (3.2) by linearity implies that $\bar{c}$ is ad-nilpotent of index at most p^s. □

We can now finish the proof of Proposition 3.5. Lemmas 3.6 and 3.7 show that $L_p(H)$ satisfies the hypotheses of Zelmanov's Theorem 2.1, by which $L_p(H)$ is nilpotent. □

Proof of Theorem 3.1. By Lemma 3.2 the quotient G/H is nilpotent. Being finitely generated, then G/H is a group of finite rank. Here, the rank of a pro-p group is the supremum of the minimum number of (topological) generators over all open subgroups.

The nilpotency of the Lie algebra $L_p(H)$ of the finitely generated pro-p group H established in Proposition 3.5 implies that H is a p-adic analytic group. This result goes back to Lazard [11]; see also [15, Corollary D]. By the Lubotzky–Mann theorem [13], being a p-adic analytic group is equivalent to being a pro-p group of finite rank. Thus, both H and G/H have finite rank, and therefore the whole pro-p group G has finite rank. Hence G is a p-adic analytic group and therefore a linear group. By Gruenberg's theorem [4], right Engel elements of a linear group are contained in the hypercentre. Since H is generated by right Engel elements, we obtain that H is contained in the hypercentre of G, and since G/H is nilpotent, the whole group G is hypercentral. Being also finitely generated, then G is nilpotent (see [14, 12.2.4]). □

4. Uniform bound for the nilpotency class

In the main Theorem 1.1, we need to prove that if a finitely generated profinite group G admits a coprime automorphism φ of prime order q all of whose fixed points are right Engel in G, then G is nilpotent. We already know that G is pronilpotent, and every Sylow p-subgroup of G is nilpotent by Theorem 3.1. This would imply the nilpotency of G if we had a uniform bound for the nilpotency class of Sylow p-subgroups independent of p. However, the nilpotency class furnished by the proof of Theorem 3.1 depends on p.

In this section we prove that for large enough primes p the nilpotency classes of Sylow p-subgroups of G are uniformly bounded above in terms of certain parameters of the group G. Together with bounds depending on p given by Theorem 3.1, this will complete the proof of the nilpotency of G. In the proof, we do not specify the conditions on p beforehand, but proceed with our arguments noting along that our conclusions hold for all large enough primes p.

One of the aforementioned parameters is the finite number of generators of G, say, d. Clearly, every Sylow p-subgroup of G can also be generated by d elements, being a homomorphic image of G by the Cartesian product of all other Sylow subgroups.

Lemma 4.1. *There are positive integers n and N_1 such that for every $p > N_1$ all fixed points of φ in the Sylow p-subgroup P of G are right n-Engel elements of P.*

Proof. In the group $C_G(\varphi) \times G$, the subsets

$$E_i = \{(x, y) \in C_G(\varphi) \times G \mid [x, {}_i y] = 1\}$$

are closed in the product topology. By hypothesis,

$$\bigcup_i E_i = C_G(\varphi) \times G.$$

Hence, by the Baire category theorem [8, Theorem 34], one of these subsets E_n contains an open subset of $C_G(\varphi) \times G$, which means that there are cosets x_0K and y_0L of open subgroups $K \leqslant C_G(\varphi)$ and $L \leqslant G$ such that $[x, {}_n y] = 1$ for all $x \in x_0K$ and $y \in y_0L$. Since the indices $|C_G(\varphi) : K|$ and $|G : L|$ are finite, for all large enough primes $p > N_1$ the Sylow p-subgroups of $C_G(\varphi)$ and G are contained in the subgroups K and L, respectively. Then for every prime $p > N_1$, in the Sylow p-subgroup P the centralizer $C_P(\varphi)$ consists of right n-Engel elements of P. □

Lemma 4.2. *There are positive integers c and N_2 such that for every $p > N_2$ in the Sylow p-subgroup P the fixed-point subgroup $C_P(\varphi)$ is nilpotent of class c.*

Proof. By Lemma 4.1, for $p > N_1$ in the Sylow p-subgroup P the subgroup $C_P(\varphi)$ is an n-Engel group. By a theorem of Burns and Medvedev [3], then $C_P(\varphi)$ has a normal subgroup N_p of exponent $e(n)$ such that the quotient $C_P(\varphi)/N_p$ is nilpotent of class $c(n)$, for some numbers $e(n)$ and $c(n)$ depending only on n. Clearly, $N_p = 1$ for all large enough primes $p > N_2 \geqslant N_1$. Thus, for every prime $p > N_2$ the subgroup $C_P(\varphi)$ is nilpotent of class $c = c(n)$. □

The following proposition will complete the proof of the main Theorem 1.1 in view of Lemmas 4.1 and 4.2.

Proposition 4.3. *There are functions $N_3(d, q, n, c)$ and $f(d, q, n, c)$ of four positive integer variables d, q, n, c with the following property. Let p be a prime, and suppose that P is a d-generated pro-p group admitting an automorphism φ of prime order $q \neq p$ such that $C_P(\varphi)$ is nilpotent of class c and consists of right n-Engel elements of P. If $p > N_3(d, q, n, c)$, then the group P is nilpotent of class at most $f(d, q, n, c)$.*

Proof. It is sufficient to obtain a bound for the nilpotency class in terms of d, q, n, c for every finite quotient T of P by a φ-invariant open normal subgroup. Consider the associated Lie ring $L(T)$ constructed with respect to the filtration consisting of the terms $\gamma_i(T)$ of the lower central series of T:

$$L(T) = \bigoplus \gamma_i(T)/\gamma_{i+1}(T).$$

As is well known, this Lie ring is nilpotent of exactly the same nilpotency class as T (see, for example, [9, Theorem 6.9]). Therefore it is sufficient to obtain a required bound for the nilpotency class of $L(T)$. We set $L = L(T)$ for brevity. Let $\tilde{L} = L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ be the Lie ring obtained by extending the ground ring by a primitive q-th root of unity ω. We regard L as $L \otimes 1$ embedded in $\tilde{L}$. The automorphism of L and of $\tilde{L}$ induced by φ is denoted by the same letter. Since the order of the automorphism φ is coprime to the orders of elements of the additive group of $\tilde{L}$, which is a p-group, we have the decomposition into analogues of eigenspaces

$$\tilde{L} = \bigoplus_{i=0}^{q-1} L_j, \qquad \text{where} \quad L_j = \{x \in \tilde{L} \mid x^{\varphi} = \omega^j x\}.$$

For clarity we call the additive subgroups L_j *eigenspaces*, and their elements *eigenvectors*. This decomposition can also be viewed as a $(\mathbb{Z}/q\mathbb{Z})$-grading of $\tilde{L}$, since

$$[L_i, L_j] \subseteq L_{i+j\,(\mathrm{mod}\,q)}.$$

Note that $L_0 = C_L(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$.

Lemma 4.4. *The fixed-point subring $C_{\tilde{L}}(\varphi)$ is nilpotent of class at most c.*

Proof. Since φ is a coprime automorphism of T, we have

$$C_L(\varphi) = \bigoplus_i (C_T(\varphi) \cap \gamma_i(T))\gamma_{i+1}(T)/\gamma_{i+1}(T).$$

Since the fixed-point subgroup $C_T(\varphi)$ is nilpotent of class c, the definition of the Lie products implies that the same is true for $C_L(\varphi)$ and therefore also for $C_{\tilde{L}}(\varphi) = C_L(\varphi) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. □

Our main aim is to enable an application to $\tilde{L}$ of the effective version of Zelmanov's theorem given by Proposition 2.3. For that we need a φ-invariant set of generators of $\tilde{L}$ such that all commutators in these generators are ad-nilpotent of bounded index.

Let $L_{(k)} = \gamma_k(T)/\gamma_{k+1}(T)$ denote the homogeneous component of weight k of the Lie ring L, and let $\tilde{L}_{(k)} = L_{(k)} \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. For clarity we say that elements of $\tilde{L}_{(k)}$ or $L_{(k)}$ are *homogeneous.* The component $L_{(1)}$ generates the Lie ring L, and $\tilde{L}_{(1)}$ generates $\tilde{L}$. If elements $t_1, \dots, t_d$ generate the group T, then their images $\bar{t}_1, \dots, \bar{t}_d$ in $L_{(1)} = T/\gamma_2(T)$ generate the Lie ring L, as well as $\tilde{L}$ (over the extended ground ring). Writing $\bar{t}_i = \sum_{j=0}^{q-1} t_{ij}$, where $t_{ij} \in \tilde{L}_{(1)} \cap L_j$ we obtain a φ-invariant set of generators of $\tilde{L}$

$$\{\omega^k t_{ij} \mid i = 1, \dots, d;\ j = 0, \dots, q-1;\ k = 0, \dots, q-1\}.$$

We claim that for $p > n$ all commutators in these generators are ad-nilpotent of index bounded in terms of q, n, c.

We set for brevity $\tilde{L}_{(v)k} = \tilde{L}_{(v)} \cap L_k$ for any weight v. A commutator of weight v in the eigenvectors t_{ij} is an eigenvector belonging to $\tilde{L}_{(v)k}$, where k is the modulo q sum of the second indices of the t_{ij} involved. We actually prove that for $p > n$ any homogeneous eigenvector $l_k \in \tilde{L}_{(v)k}$ is ad-nilpotent of index s bounded in terms of q, n, c. It is clearly sufficient to show that $[x_j, {}_s l_k] = 0$ for any homogeneous eigenvector $x_j \in \tilde{L}_{(u)j}$, for any weights u, v and any indices $j, k \in \{0, 1, \dots, q-1\}$. (Here we use indices j, k for elements x_j, l_k only to indicate the eigenspaces they belong to.) First we consider the case where $j = 0$.

Lemma 4.5. *If $p > n$, then for any weights u, v, for any eigenvector $x_0 \in \tilde{L}_{(u)0}$ and any homogeneous element $l \in \tilde{L}_{(v)}$ we have $[x_0, {}_n l] = 0$.*

Proof. Since φ is an automorphism of coprime order, for $x_0 \in \tilde{L}_{(u)0}$ there are elements $y_i \in C_T(\varphi) \cap \gamma_u(T)$ such that $x_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$, where $\bar{y}_i$ is the image of y_i in $\gamma_u(T)/\gamma_{u+1}(T)$ (here the indices of the y_i are used for numbering). For any $\bar{h} \in L_{(v)}$, there is an element $h \in T \cap \gamma_v(T)$ such that $\bar{h}$ is the image of h in $L_{(v)} = \gamma_v(T)/\gamma_{v+1}(T)$. Since $[y_i, {}_n h] = 1$ in the group T by the hypothesis of Proposition 4.3, we have $[\bar{y}_i, {}_n \bar{h}] = 0$ in L for every i. Hence, by linearity,

$$[x_0, {}_n \bar{h}] = 0 \tag{4.1}$$

in $\tilde{L}$. Note, however, that $\tilde{L}_{(v)}$ does not consist only of $\mathbb{Z}[\omega]$-multiples of elements of $L_{(v)}$. Nevertheless, (4.1) looks like the n-Engel identity, which implies its linearization, which in turn survives extension of the ground ring, and then implies the required property due to the condition $p > n$ making $n!$ an invertible element of the ground ring. However, we cannot simply make a reference to these well-known facts, since this is not exactly an identity, so we reproduce these familiar arguments in our specific situation (jumping over one of the steps).

We substitute $a_1 + \dots + a_n$ for $\bar{h}$ in (4.1) with arbitrary homogeneous elements $a_i \in L_{(v)}$ (the indices of the a_i are used for numbering). Thus,

$$\big[x_0,\ {}_n(a_1 + \dots + a_n)\big] = 0$$

for any elements $a_i \in L_{(v)}$, some of which may also be equal to one another. After expanding all brackets, we obtain the equation

$$0 = \big[x_0,\ {}_n(a_1 + \cdots + a_n)\big] = \sum_{\substack{i_1 \geqslant 0, \ldots, i_n \geqslant 0\\ i_1 + \cdots + i_n = n}} \varkappa_{i_1,\ldots,i_n}, \tag{4.2}$$

where $\varkappa_{i_1,\ldots,i_n}$ denotes the sum of all commutators of degree i_j in a_j. Replacing a_1 with 0 (only this formal occurrence, keeping intact all other a_i even if some are equal to a_1) shows that

$$0 = \sum_{\substack{i_1=0,\, i_2 \geqslant 0, \ldots, i_n \geqslant 0\\ i_1 + \cdots + i_n = n}} \varkappa_{i_1,\ldots,i_n}.$$

Hence we can remove from the right-hand side of (4.2) all terms not involving a_1 as a formal entry (keeping the other a_i even if some are equal to a_1). We obtain

$$0 = \big[x_0, {}_n(a_1 + \cdots + a_n)\big] = \sum_{\substack{i_1 \geqslant 1,\, i_2 \geqslant 0, \ldots, i_n \geqslant 0\\ i_1 + \cdots + i_n = n}} \varkappa_{i_1,\ldots,i_n}.$$

Then we do the same with a_2 for the resulting equation, and so on, consecutively with all the a_i. As a result we obtain

$$0 = \sum_{\substack{i_1 \geqslant 1, \ldots, i_n \geqslant 1\\ i_1 + \cdots + i_n = n}} \varkappa_{i_1,\ldots,i_n} = \varkappa_{1,\ldots,1},$$

that is,

$$0 = \sum_{\pi \in S_n} \big[x_0,\ a_{\pi(1)}, \ldots, a_{\pi(n)}\big], \tag{4.3}$$

where the right-hand side is the desired linearization. Every element $l \in \tilde{L}_{(v)}$ can be written as a linear combination $l = m_0 + \omega m_1 + \omega^2 m_2 + \cdots + \omega^{q-2} m_{q-2}$, where $m_i \in L_{(v)}$. Then

$$\begin{aligned}[x_0, {}_n l] &= \big[x_0,\ {}_n(m_0 + \omega m_1 + \omega^2 m_2 + \cdots + \omega^{q-2} m_{q-2})\big]\\ &= \sum_{i=0}^{n(q-2)} \omega^i \sum_{j_1 + 2j_2 + \cdots + (q-2)j_{q-2} = i} \lambda_{j_0, j_1, \ldots, j_{q-2}},\end{aligned}$$

where $\lambda_{j_0,j_1,\ldots,j_{q-2}}$ denotes the sum of all commutators in the expansion of the left-hand side with weight j_s in m_s. But each of these sums is clearly symmetric and therefore is equal to 0 as a consequence of (4.3), where, if an element a_i is required to be repeated n_i times, then the coefficient $n_i!$ appears, which is invertible in the ground ring, since $n_i < p$ and the additive group is a p-group. The lemma is proved. □

Lemma 4.6. *If $p > n$, then for any v and k, any homogeneous eigenvector $l_k \in \tilde{L}_{(v)k}$ is ad-nilpotent of index bounded in terms of q, n, c.*

Proof. First consider the case $k = 0$. Then $l_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$, where $\bar{y}_i$ is the image of an element $y_i \in C_T(\varphi) \cap \gamma_v(T)$ in $\gamma_v(T)/\gamma_{v+1}(T)$ (the indices of the y_i are used for numbering). For each i, since y_i^{-1} is a right n-Engel element of T by hypothesis, y_i is a left $(n+1)$-Engel element by a result of Heineken [6] (see also [14, 12.3.1]). For any homogeneous element $\bar{h} \in L_{(u)}$ there is an element $h \in T \cap \gamma_u(T)$ such that $\bar{h}$ is the image of h in $\gamma_u(T)/\gamma_{u+1}(T)$. Since $[h, {}_{n+1} y_i] = 1$ in the group T, we have $[\bar{h}, {}_{n+1} \bar{y}_i] = 0$ in L for every i. Hence, by linearity, each $\bar{y}_i$ is ad-nilpotent in L of index at most $n+1$. Let M be the subring of L generated by $\bar{y}_0, \bar{y}_1, \dots, \bar{y}_{q-2}$. Since $M \leqslant C_L(\varphi)$, the subring M is nilpotent of class at most c by Lemma 4.4. We can now apply Lemma 2.4, by which

$$[L, \underbrace{M, M, \dots, M}_{\varepsilon}] = 0$$

for some $\varepsilon = \varepsilon(q-1, n+1, c)$ bounded in terms of q, n, c. This equation remains valid after extension of the ground ring. In particular, $l_0 = \sum_{i=0}^{q-2} \omega^i \bar{y}_i$ is ad-nilpotent in $\tilde{L}$ of index at most $\varepsilon = \varepsilon(q-1, n+1, c)$, as required.

Now suppose that $k \neq 0$. For a homogeneous eigenvector $x_j \in \tilde{L}_{(u)j}$, the commutator

$$[x_j,\, {}_{q+n-1}l_k] = \big[x_j,\, \underbrace{l_k, \dots, l_k}_{s},\, l_k, \dots, l_k\big] \tag{4.4}$$

has an initial segment of length $s+1 \leqslant q$ that is a homogeneous eigenvector $x_0 = [x_j,\, {}_s l_k] \in \tilde{L}_{(w)0}$ (for some weight w). Indeed, the congruence $j + sk \equiv 0 \pmod{q}$ has a solution $s \in \{0, 1, \dots, q-1\}$ since $k \not\equiv 0 \pmod{q}$. There remain at least n further entries of l_k in (4.4), so that we have a subcommutator of the form $[x_0,\, {}_n l_k]$, which is equal to 0 by Lemma 4.5. Thus, by linearity, l_k is ad-nilpotent of index at most $q+n-1$. □

We now finish the proof of Proposition 4.3. By Lemma 4.6, for $p > n$ every commutator in the generators t_{ij} of the Lie ring $\tilde{L}$ is ad-nilpotent of index bounded in terms of q, n, c. The same is true for the generators in the φ-invariant set

$$\{\omega^k t_{ij} \mid i = 1, \dots, d;\ j = 0, \dots, q-1;\ k = 0, \dots, q-1\},$$

which consists of $q^2 d$ elements. The fixed-point subring $C_{\tilde{L}}(\varphi)$ is nilpotent of class at most c by Lemma 4.4. Thus, for $p > n$ the Lie ring $\tilde{L}$ and its group of automorphisms $\langle\varphi\rangle$ satisfy the hypotheses of Proposition 2.3. By this proposition, there exist positive integers e and r depending only on d, q, n, c such that $e\gamma_r(\tilde{L}) = 0$. The additive group of $\tilde{L}$ is a p-group. Therefore, if $p > e$, then e is invertible in the ground ring, so that we obtain $\gamma_r(L) = 0$. It remains to put $N_3(d,q,n,c) = \max\{n, e\}$ and $f(d,q,n,c) = r-1$.

We thus proved that for $p > N_3(d,q,n,c)$ every finite quotient of P by a φ-invariant normal open subgroup is nilpotent of class at most $f(d,q,n,c)$. Therefore P is nilpotent of class at most $f(d,q,n,c)$ if $p > N_3(d,q,n,c)$. □

We finally combine all the results in the proof of the main theorem.

Proof of Theorem 1.1. Recall that G is a profinite group admitting a coprime automorphism φ of prime order q all of whose fixed points are right Engel in G; we need to prove that G is locally nilpotent. Any finite set $S \subseteq G$ is contained in the φ-invariant finite set $S^{\langle\varphi\rangle} = \{s^{\varphi^k} \mid s \in S,\ k = 0, 1, \dots, q-1\}$. Therefore we can assume that the group G is finitely generated, say, by d elements, and then need to prove that G is nilpotent. As noted in the Introduction, the group G is pronilpotent, so we only need to prove that all Sylow p-subgroups of G are nilpotent of class bounded by some number independent of p.

Let n and N_1 be the numbers given by Lemma 4.1, and c and N_2 the numbers given by Lemma 4.2. Further, let $N_3(d,q,n,c)$ be the number given by Proposition 4.3. Then for every prime $p > \max\{N_1, N_2, N_3(d,q,n,c)\}$ the Sylow p-subgroup of G is nilpotent of class at most $f(d,q,n,c)$ for the function given by Proposition 4.3. Since every Sylow p-subgroup is nilpotent by Theorem 3.1, we obtain a required uniform bound for the nilpotency classes of Sylow p-subgroups independent of p. □

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