EFFECT OF NON-POLYNOMIAL INPUT TO A SWITCHING CIRCUIT

Ling Wing Kuen, and Tam Kwong Shun, Peter

Department of Electronic and Information Engineering
The Hong Kong Polytechnic University
Hung Hom, Kowloon, Hong Kong
Hong Kong Special Administrative Region, China
Tel: (852) 2766-6252, Fax: (852) 2362-8439
Email: bingo@encserver.eie.polyu.edu.hk

Abstract: In this paper, the validity of the state-space averaging method is analyzed. We assume that the state-space piecewise method is an exact model for a fast switching circuit. Based on this model, we compute the error predicted by the state-space averaging method. It is found that the error for a polynomial input is bounded by two polynomials with the same order as that of the input. And the percentage error is bounded by a constant. Hence, if the acceptable level is within that constant, then the state-space averaging method can be applied. Similar analysis is carried out on a non-polynomial input. A sinusoidal function is chosen because of its wide applications on AC circuits. Although a similar result is obtained, the percentage error for the sinusoidal input is much greater than that of the polynomial input. Hence, the state-space averaging method may not be so good for the AC analysis.

Keywords
State-space piecewise method, state-space averaging method, polynomial, sinusoidal function

I. INTRODUCTION

Switching circuits are playing an increasingly important role in power electronics in this decade [1]. Since they are non-linear and time-varying in nature due to the non-zero initial energy stored in the circuit elements and the changing status of switches, some traditional methods, such as impulse response and frequency response, cannot be readily applied to analyze them.

In order to work on this problem, a state-space averaging method has been proposed [2]. There are lots of advantages for employing this method. Since it involves only a first order differential vector equation, the state vector function and the output function can be computed implicitly at any instant easily. Also, as the computation complexity is low, it is good for simulation. However, the prediction error introduced by it may cause some faults in analysis and design.

A state-space piecewise method has been proposed for analyzing the switching circuits [3]. However, iterations are required to compute the state vector function and the output function. It takes a very long time to reach the steady state and the simulation complexity is very high.

Some numerical methods have been proposed to speed up the state-space piecewise method [4]. However, the assumptions made and the approximations taken are sometimes invalid and inappropriate, and these may lead to significant prediction error.

In this paper, the state-space averaging method and the state-space piecewise method are reviewed in section II and section III, respectively. The errors predicted by the state-space averaging method for the polynomial input and the sinusoidal input are discussed in section IV and section V, respectively. Finally, simulation results and concluding remarks are given in section VI and section VII, respectively.

II. REVIEW ON STATE-SPACE AVERAGING METHOD

Assume that a switching circuit consists of two topologies, topology I and topology II, and the duty cycle at each topology is *50%*. If $\mathbf{A_1}$, $\mathbf{B_1}$, $\mathbf{C_1}$ and $\mathbf{D_1}$ are the matrices of a state-space representation of the circuit at the topology I and $\mathbf{A_2}$, $\mathbf{B_2}$, $\mathbf{C_2}$ and $\mathbf{D_2}$ are that at the topology II, then the 'average' state-space representation of the whole circuit is:

$$A=\frac{A_1+A_2}{2},B=\frac{B_1+B_2}{2},C=\frac{C_1+C_2}{2},D=\frac{D_1+D_2}{2} \qquad \textbf{(1).}$$

The state vector function and the output function of the system are:

$$x(t)=e^{(t-t_0)\cdot A}\cdot x(t_0)+\int_{t_0}^{t}e^{(t-\tau)\cdot A}\cdot B\cdot u(\tau)d\tau \textbf{ and}$$

$$y(t)=C\cdot e^{(t-t_0)\cdot A}\cdot x(t_0)+C\cdot\int_{t_0}^{t}e^{(t-\tau)\cdot A}\cdot B\cdot u(\tau)d\tau+D\cdot u(t) \qquad \textbf{(2),}$$

respectively, for $t\geq t_0$, where $\boldsymbol{x(t_0)}$ is the initial condition of the state vector at $t=t_0$ and $u(t)$ is the input of the system. For simplicity, assume that $\mathbf{A}$ is an NxN matrix with distinct eigenvalues λ_i, for $i=1,2,\ldots,N$, respectively. Apply the Cayley Hamilton expansion to the matrix exponential terms in equation (2), that is:

$$e^{t\cdot A}=\sum_{i=1}^{N}\alpha_i(t)\cdot A^{i-1} \qquad \textbf{(3),}$$

where

$$\begin{bmatrix}\alpha_1(t)\\ \vdots\\ \alpha_N(t)\end{bmatrix}=\begin{bmatrix}1 & \cdots & \lambda_1^{N-1}\\ \vdots & \ddots & \vdots\\ 1 & \cdots & \lambda_N^{N-1}\end{bmatrix}^{-1}\cdot\begin{bmatrix}e^{\lambda_1\cdot t}\\ \vdots\\ e^{\lambda_N\cdot t}\end{bmatrix}=\begin{bmatrix}a_{11} & \cdots & a_{1N}\\ \vdots & \ddots & \vdots\\ a_{N1} & \cdots & a_{NN}\end{bmatrix}\cdot\begin{bmatrix}e^{\lambda_1\cdot t}\\ \vdots\\ e^{\lambda_N\cdot t}\end{bmatrix} \textbf{(4).}$$

Then the state vector function and the output function become:

$$x(t)=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\lambda_i\cdot(t-t_0)}\cdot A^{j-1}\cdot x(t_0)+$$
$$\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\int_{t_0}^{t}e^{\lambda_i\cdot(t-\tau)}\cdot u(\tau)d\tau \textbf{ and}$$
$$y(t)=C\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\lambda_i\cdot(t-t_0)}\cdot A^{j-1}\cdot x(t_0)$$
$$+C\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\int_{t_0}^{t}e^{\lambda_i\cdot(t-\tau)}\cdot u(\tau)d\tau+D\cdot u(t)\quad \textbf{(5)},$$

respectively. If the eigenvalues are not distinct, similar results are obtained.

III. REVIEW ON STATE-SPACE PIECEWISE METHOD

For simplicity, under the same assumptions made in section II, if $\mathbf{A_1}$ and $\mathbf{A_2}$ are NxN matrices with distinct eigenvalues λ_i and λ_i', for $i=1,2,\ldots,N$, respectively, then the state vector function and the output function are:

$$x(t)=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\lambda_i\cdot(t-t_0-n\cdot T_s)}\cdot A_1^{\,j-1}\cdot x(t_0+n\cdot T_s)$$
$$+\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A_1^{\,j-1}\cdot B_1\cdot\int_{t_0+n\cdot T_s}^{t}e^{\lambda_i\cdot(t-\tau)}\cdot u(\tau)d\tau \textbf{ and}$$
$$y(t)=C_1\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\lambda_i\cdot(t-t_0-n\cdot T_s)}\cdot A_1^{\,j-1}\cdot x(t_0+n\cdot T_s)$$
$$+C_1\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A_1^{\,j-1}\cdot B_1\cdot\int_{t_0+n\cdot T_s}^{t}e^{\lambda_i\cdot(t-\tau)}\cdot u(\tau)d\tau+D_1\cdot u(t)\textbf{(6)},$$

respectively, for $t_0+n\cdot T_s\le t\le t_0+(n+0.5)\cdot T_s$, and

$$x(t)=\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot e^{\lambda_i'\left(t-t_0-\left(n+\frac{1}{2}\right)T_s\right)}\cdot A_2^{\,j-1}\cdot x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)$$
$$+\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot A_2^{\,j-1}\cdot B_2\cdot\int_{t_0+\left(n+\frac{1}{2}\right)T_s}^{t}e^{\lambda_i'\cdot(t-\tau)}\cdot u(\tau)d\tau \textbf{ and}$$
$$y(t)=C_2\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot e^{\lambda_i'\left(t-t_0-\left(n+\frac{1}{2}\right)T_s\right)}\cdot A_2^{\,j-1}\cdot x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)$$
$$+C_2\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot A_2^{\,j-1}\cdot B_2\cdot\int_{t_0+\left(n+\frac{1}{2}\right)T_s}^{t}e^{\lambda_i'\cdot(t-\tau)}\cdot u(\tau)d\tau+D_2\cdot u(t)\textbf{(7)},$$

respectively, for $t_0+(n+0.5)\cdot T_s\le t\le t_0+(n+1)\cdot T_s$, where T_s is the switching period, $n=0,1,2,\ldots$,

$$e^{t\cdot A_1}=\sum_{i=1}^{N}\alpha_i(t)\cdot A_1^{\,i-1},$$
$$e^{t\cdot A_2}=\sum_{i=1}^{N}\alpha_i'(t)\cdot A_2^{\,i-1}\qquad \textbf{(8)},$$

and

$$\begin{bmatrix}\alpha_1(t)\\ \vdots\\ \alpha_N(t)\end{bmatrix}=\begin{bmatrix}\mathbf{1} & \cdots & \lambda_1^{\,N-1}\\ \vdots & \ddots & \vdots\\ \mathbf{1} & \cdots & \lambda_N^{\,N-1}\end{bmatrix}^{-1}\cdot\begin{bmatrix}e^{\lambda_1\cdot t}\\ \vdots\\ e^{\lambda_N\cdot t}\end{bmatrix}=\begin{bmatrix}a_{11} & \cdots & a_{1N}\\ \vdots & \ddots & \vdots\\ a_{N1} & \cdots & a_{NN}\end{bmatrix}\cdot\begin{bmatrix}e^{\lambda_1\cdot t}\\ \vdots\\ e^{\lambda_N\cdot t}\end{bmatrix},$$

$$\begin{bmatrix}\alpha_1'(t)\\ \vdots\\ \alpha_N'(t)\end{bmatrix}=\begin{bmatrix}\mathbf{1} & \cdots & \lambda_1'^{\,N-1}\\ \vdots & \ddots & \vdots\\ \mathbf{1} & \cdots & \lambda_N'^{\,N-1}\end{bmatrix}^{-1}\cdot\begin{bmatrix}e^{\lambda_1'\cdot t}\\ \vdots\\ e^{\lambda_N'\cdot t}\end{bmatrix}=\begin{bmatrix}a'_{11} & \cdots & a'_{1N}\\ \vdots & \ddots & \vdots\\ a'_{N1} & \cdots & a'_{NN}\end{bmatrix}\cdot\begin{bmatrix}e^{\lambda_1'\cdot t}\\ \vdots\\ e^{\lambda_N'\cdot t}\end{bmatrix}\textbf{(9)}.$$

IV. ERROR PREDICTED BY THE STATE-SPACE AVERGING METHOD FOR A POLYNOMIAL INPUT

There are many applications of building a switching circuit using a polynomial input. For example, a DC-DC converter is a switching circuit using the step input [5]. It can be seen from equation (5) that the input affects the output only through the following integration term:

$$\int_{t_0}^{t}e^{\lambda_i\cdot(t-\tau)}\cdot u(\tau)d\tau\qquad \textbf{(10)}.$$

State-space averaging method

If the input is a polynomial function of time, then we can compute the integral of (10) at the switching instants using the following formula:

$$\left.\int_{t_0}^{t}e^{\lambda_i\cdot(t-\tau)}\cdot\tau^k d\tau\right|_{t=t_0+n\cdot T_s}$$
$$=\sum_{m=0}^{k}\frac{k!}{m!\cdot\lambda_i^{\,k-m+1}}\cdot\left[t_0^{\,m}\cdot e^{\lambda_i\cdot n\cdot T_s}-(t_0+n\cdot T_s)^m\right]\textbf{, for } k\ge\mathbf{0}\quad\textbf{(11)}.$$

By substituting equation (11) into equation (5), the state vector function and the output function at the switching instants are:

$$x(t_0+n\cdot T_s)=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\lambda_i\cdot n\cdot T_s}\cdot A^{j-1}\cdot x(t_0)$$
$$+\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\left\{\sum_{m=0}^{k}\frac{k!}{m!\cdot\lambda_i^{\,k-m+1}}\cdot\left[t_0^{\,m}\cdot e^{\lambda_i\cdot n\cdot T_s}-(t_0+n\cdot T_s)^m\right]\right\}\textbf{ and}$$
$$y(t_0+n\cdot T_s)=C\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\lambda_i\cdot n\cdot T_s}\cdot A^{j-1}\cdot x(t_0)$$
$$+C\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\left\{\sum_{m=0}^{k}\frac{k!}{m!\cdot\lambda_i^{\,k-m+1}}\cdot\left[t_0^{\,m}\cdot e^{\lambda_i\cdot n\cdot T_s}-(t_0+n\cdot T_s)^m\right]\right\}$$
$$+D\cdot(t_0+n\cdot T_s)^k\qquad\textbf{(12)},$$

respectively.

Although equation (12) appears to be quite complicated, the state vector can be written as the sum of the zero-input response $F_0(n)\cdot x(t_0)$ and the zero-state response $F_1(n)\cdot g_0(t_0)+F_2(n)\cdot g_1(t_0+n\cdot T_s)$, as follows:

$$x(t_0+n\cdot T_s)=F_0(n)\cdot x(t_0)+F_1(n)\cdot g_0(t_0)+F_2(n)\cdot g_1(t_0+n\cdot T_s)\textbf{(13)},$$

where $g_0(t_0)$ and $g_1(t_0+n\cdot T_s)$ correspond to the k^{th} order polynomial functions of t_0 and $t_0+n\cdot T_s$, respectively. And the output is:

$$y(t_0+n\cdot T_s)=F_0'(n)\cdot x(t_0)+F_1'(n)\cdot g_0(t_0)+F_2'(n)\cdot g_1(t_0+n\cdot T_s)\ \textbf{(14)}.$$

State-space piecewise method

Similar to the state-space averaging method, the corresponding integration terms at the switching instants can be computed by the following formulae:

$$\left.\int_{t_0+n\cdot T_s}^{t} e^{\lambda_i\cdot(t-\tau)}\cdot\tau^k d\tau\right|_{t=t_0+\left(n+\frac{1}{2}\right)\cdot T_s}$$

$$=\sum_{m=0}^{k}\frac{k!}{m!\cdot\lambda_i^{\ k-m+1}}\cdot\left[\left(t_0+n\cdot T_s\right)^m\cdot e^{\frac{\lambda_i\cdot T_s}{2}}-\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)^m\right]\text{, for } k\geq 0\text{, and}$$

$$\left.\int_{t_0+\left(n+\frac{1}{2}\right)\cdot T_s}^{t} e^{\lambda_i'\cdot(t-\tau)}\cdot\tau^k d\tau\right|_{t=t_0+(n+1)\cdot T_s}=$$

$$\sum_{m=0}^{k}\frac{k!}{m!\cdot\lambda_i'^{\ k-m+1}}\cdot\left[\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)^m\cdot e^{\frac{\lambda_i'\cdot T_s}{2}}-\left(t_0+(n+1)\cdot T_s\right)^m\right]\text{, for } k\geq 0 \quad \textbf{(15),}$$

respectively. The state vector function and the output function at the switching instants are:

$$x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)=R\cdot x\left(t_0+n\cdot T_s\right)+\sum_{m=0}^{k}\left[P_m\cdot\left(t_0+n\cdot T_s\right)^m-Q_m\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)^m\right],$$

$$x\left(t_0+(n+1)\cdot T_s\right)=R'\cdot x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)+\sum_{m=0}^{k}\left[P_m'\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)^m-Q_m'\cdot\left(t_0+(n+1)\cdot T_s\right)^m\right],$$

$$y\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)=C_2\cdot R\cdot x\left(t_0+n\cdot T_s\right)+C_2\cdot\sum_{m=0}^{k}\left[P_m\cdot\left(t_0+n\cdot T_s\right)^m-Q_m\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)^m\right]+D_2\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)^k\ \text{and}$$

$$y\left(t_0+(n+1)\cdot T_s\right)=C_1\cdot R'\cdot x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)+C_1\cdot\sum_{m=0}^{k}\left[P_m'\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)^m-Q_m'\cdot\left(t_0+(n+1)\cdot T_s\right)^m\right]+D_1\cdot\left(t_0+(n+1)\cdot T_s\right)^k \quad \textbf{(16),}$$

respectively, where

$$P_m=\sum_{j=1}^{N}\sum_{i=1}^{N}A_1^{\ j-1}\cdot B_1\cdot a_{ji}\cdot\frac{k!}{m!\cdot\lambda_i^{\ k+1-m}}\cdot e^{\frac{\lambda_i\cdot T_s}{2}},$$

$$Q_m=\sum_{j=1}^{N}\sum_{i=1}^{N}A_1^{\ j-1}\cdot B_1\cdot a_{ji}\cdot\frac{k!}{m!\cdot\lambda_i^{\ k+1-m}},$$

$$R=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\frac{\lambda_i\cdot T_s}{2}}\cdot A_1^{\ j-1},$$

$$P_m'=\sum_{j=1}^{N}\sum_{i=1}^{N}A_2^{\ j-1}\cdot B_2\cdot a_{ji}'\cdot\frac{k!}{m!\cdot\lambda_i'^{\ k+1-m}}\cdot e^{\frac{\lambda_i'\cdot T_s}{2}},$$

$$Q_m'=\sum_{j=1}^{N}\sum_{i=1}^{N}A_2^{\ j-1}\cdot B_2\cdot a_{ji}'\cdot\frac{k!}{m!\cdot\lambda_i'^{\ k+1-m}},$$

$$R'=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}'\cdot e^{\frac{\lambda_i'\cdot T_s}{2}}\cdot A_2^{\ j-1} \quad \textbf{(17).}$$

By solving the difference equation (16), the state vector function and the output function at the switching instants can be expressed as:

$$\begin{aligned}x\left(t_0+n\cdot T_s\right)=&\left(R'\cdot R\right)^n\cdot x\left(t_0\right)\\&+\sum_{p=0}^{n-1}\left(R'\cdot R\right)^{n-p-1}\cdot\sum_{m=0}^{k}\left(P_m'-R'\cdot Q_m\right)\cdot\left(t_0+\left(p+\frac{1}{2}\right)\cdot T_s\right)^m\\&+\sum_{p=1}^{n-1}\sum_{m=0}^{k}\left[\left(R'\cdot R\right)^{n-p-1}\cdot R'\cdot P_m-\left(R'\cdot R\right)^{n-p}\cdot Q_m'\right]\cdot\left(t_0+p\cdot T_s\right)^m\\&+\sum_{m=0}^{k}\left(R'\cdot R\right)^{n-1}\cdot R'\cdot P_m\cdot t_0^{\ m}-\sum_{m=0}^{k}Q_m'\cdot\left(t_0+n\cdot T_s\right)^m\ \text{and}\end{aligned}$$

$$\begin{aligned}y\left(t_0+n\cdot T_s\right)=&C_1\cdot\left(R'\cdot R\right)^n\cdot x\left(t_0\right)\\&+\sum_{p=0}^{n-1}C_1\cdot\left(R'\cdot R\right)^{n-p-1}\cdot\sum_{m=0}^{k}\left(P_m'-R'\cdot Q_m\right)\cdot\left(t_0+\left(p+\frac{1}{2}\right)\cdot T_s\right)^m\\&+\sum_{p=1}^{n-1}\sum_{m=0}^{k}C_1\cdot\left[\left(R'\cdot R\right)^{n-p-1}\cdot R'\cdot P_m-\left(R'\cdot R\right)^{n-p}\cdot Q_m'\right]\cdot\left(t_0+p\cdot T_s\right)^m\\&+\sum_{m=0}^{k}C_1\cdot\left(R'\cdot R\right)^{n-1}\cdot R'\cdot P_m\cdot t_0^{\ m}-\sum_{m=0}^{k}C_1\cdot Q_m'\cdot\left(t_0+n\cdot T_s\right)^m\\&+D_1\cdot\left(t_0+n\cdot T_s\right)^k\end{aligned} \quad \textbf{(18),}$$

respectively. Similarly, the state vector in equation (18) can be written as a sum of the zero-input response $E_0(n)\cdot x(t_0)$ and the zero-state response $E_1(n)\cdot f_0(t_0)+E_2(n)\cdot f_1(t_0+0.5\cdot T_s)+\ldots+E_{2\cdot n+1}(n)\cdot f_{2\cdot n}(t_0+n\cdot T_s)$, as follows:

$$x\left(t_0+n\cdot T_s\right)=E_0(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}E_{p+1}(n)\cdot f_p\left(t_0+\frac{p}{2}\cdot T_s\right) \quad \textbf{(19),}$$

where $f_p(t_0+p\cdot T_s/2)$ correspond to the k^{th} order polynomial function of $t_0+p\cdot T_s/2$. And the output is:

$$y\left(t_0+n\cdot T_s\right)=E_0'(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}E_{p+1}'(n)\cdot f_p\left(t_0+\frac{p}{2}\cdot T_s\right) \quad \textbf{(20).}$$

Error equations

From equations (13), (14), (19) and (20), the prediction errors are:

$$\Delta x\left(t_0+n\cdot T_s\right)=H_0(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}H_{p+1}(n)\cdot h_p\left(t_0+\frac{p}{2}\cdot T_s\right)\ \text{and}$$

$$\Delta y\left(t_0+n\cdot T_s\right)=H_0'(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}H_{p+1}'(n)\cdot h_p\left(t_0+\frac{p}{2}\cdot T_s\right) \quad \textbf{(21),}$$

respectively. As $h_p(t_0+t\cdot T_s/2)$ is a polynomial with the same order as that of the input, the error is bounded by two characteristic polynomials of the same order as that of the input.

The percentage errors are:

$$\frac{\Delta x\left(t_0+n\cdot T_s\right)}{x\left(t_0+n\cdot T_s\right)}=\frac{H_0(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}H_{p+1}(n)\cdot h_p\left(t_0+\frac{p}{2}\cdot T_s\right)}{E_0(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}E_{p+1}(n)\cdot f_p\left(t_0+\frac{p}{2}\cdot T_s\right)}\ \text{and}$$

$$\frac{\Delta y\left(t_0+n\cdot T_s\right)}{y\left(t_0+n\cdot T_s\right)}=\frac{H_0'(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}H_{p+1}'(n)\cdot h_p\left(t_0+\frac{p}{2}\cdot T_s\right)}{E_0'(n)\cdot x\left(t_0\right)+\sum_{p=0}^{2\cdot n}E_{p+1}'(n)\cdot f_p\left(t_0+\frac{p}{2}\cdot T_s\right)} \quad \textbf{(22),}$$

respectively. Since the numerator and the denominator are of the same order, the steady-state value is bounded by a constant.

V. EFFECT OF NON-POLYNOMIAL INPUT

In many cases, the input cannot be assumed to be a polynomial function of time. The most common type of a non-polynomial input is a periodic signal, which is made up of harmonically related sinusoids. Hence, the sinusoidal input is addressed in this paper.

State-space averaging method

The integral of (10) can be computed by:

$$\int_{t_0}^{t} e^{\lambda_i\cdot(t-\tau)}\cdot\sin(\omega\cdot\tau)d\tau\Bigg|_{t_0+n\cdot T_s} = \frac{\sin(\omega\cdot t_0)\cdot e^{\lambda_i\cdot n\cdot T_s}-\sin(\omega\cdot(t_0+n\cdot T_s))}{\lambda_i\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)} + \frac{\omega\cdot\cos(\omega\cdot t_0)\cdot e^{\lambda_i\cdot n\cdot T_s}-\omega\cdot\cos(\omega\cdot(t_0+n\cdot T_s))}{\lambda_i^2\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)} \quad \textbf{(23).}$$

The state vector function and the output function computed at the switching instants are:

$$\begin{aligned} x(t_0+n\cdot T_s) &= S_0(n)\cdot x(t_0) \\ &+T_0(n)\cdot\sin(\omega\cdot t_0)+T_0'(n)\cdot\cos(\omega\cdot t_0) \\ &-U_0\cdot\sin(\omega\cdot(t_0+n\cdot T_s))-U_0'\cdot\cos(\omega\cdot(t_0+n\cdot T_s)) \ \textbf{and} \\ y(t_0+n\cdot T_s) &= C\cdot S_0(n)\cdot x(t_0) \\ &+C\cdot T_0(n)\cdot\sin(\omega\cdot t_0)+C\cdot T_0'(n)\cdot\cos(\omega\cdot t_0) \\ &-C\cdot U_0\cdot\sin(\omega\cdot(t_0+n\cdot T_s))-C\cdot U_0'\cdot\cos(\omega\cdot(t_0+n\cdot T_s)) \\ &+D\cdot\sin(\omega\cdot(t_0+n\cdot T_s)) \qquad \textbf{(24),} \end{aligned}$$

respectively, where

$$S_0(n)=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\lambda_i\cdot n\cdot T_s}\cdot A^{j-1},$$

$$T_0(n)=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\frac{e^{\lambda_i\cdot n\cdot T_s}}{\lambda_i\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)},$$

$$T_0'(n)=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\frac{\omega\cdot e^{\lambda_i\cdot n\cdot T_s}}{\lambda_i^2\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)},$$

$$U_0=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\frac{1}{\lambda_i\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)},$$

$$U_0'=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A^{j-1}\cdot B\cdot\frac{\omega}{\lambda_i^2\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)} \quad \textbf{(25).}$$

The state vector function can be realized as the sum of the zero input response $S_0(n)\cdot x(t_0)$ and the zero state response as $T_0(n)\cdot\sin(\omega\cdot t_0)+T'_0(n)\cdot\cos(\omega\cdot t_0)-U_0\cdot\sin(\omega\cdot(t_0+n\cdot T_s))-U'_0\cdot\cos(\omega\cdot(t_0+n\cdot T_s))$.

State-space piecewise method

The integral of (10) can be computed by:

$$\int_{t_0+n\cdot T_s}^{t} e^{\lambda_i\cdot(t-\tau)}\cdot\sin(\omega\cdot\tau)d\tau\Bigg|_{t_0+\left(n+\frac{1}{2}\right)\cdot T_s} = \frac{\sin(\omega\cdot(t_0+n\cdot T_s))\cdot e^{\frac{\lambda_i\cdot T_s}{2}}-\sin\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)}{\lambda_i\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)} + \frac{\omega\cdot\cos(\omega\cdot(t_0+n\cdot T_s))\cdot e^{\frac{\lambda_i\cdot T_s}{2}}-\omega\cdot\cos\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)}{\lambda_i^2\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)} \ \textbf{and}$$

$$\int_{t_0+\left(n+\frac{1}{2}\right)\cdot T_s}^{t} e^{\lambda_i\cdot(t-\tau)}\cdot\sin(\omega\cdot\tau)d\tau\Bigg|_{t_0+(n+1)\cdot T_s} = \frac{\sin\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)\cdot e^{\frac{\lambda_i\cdot T_s}{2}}-\sin(\omega\cdot(t_0+(n+1)\cdot T_s))}{\lambda_i\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)} + \frac{\omega\cdot\cos\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)\cdot e^{\frac{\lambda_i\cdot T_s}{2}}-\omega\cdot\cos(\omega\cdot(t_0+(n+1)\cdot T_s))}{\lambda_i^2\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)} \quad \textbf{(26),}$$

respectively, the state vector function and the output function computed at the switching instant are:

$$\begin{aligned} x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right) &= S_1\cdot x(t_0+n\cdot T_s) \\ &+T_1\cdot\sin(\omega\cdot(t_0+n\cdot T_s))+T_2\cdot\cos(\omega\cdot(t_0+n\cdot T_s)) \\ &-U_1\cdot\sin\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)-U_2\cdot\cos\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right), \\ x(t_0+(n+1)\cdot T_s) &= S_2\cdot x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right) \\ &+T_1'\cdot\sin\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)+T_2'\cdot\cos\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right) \\ &-U_1'\cdot\sin(\omega\cdot(t_0+(n+1)\cdot T_s))-U_2'\cdot\cos(\omega\cdot(t_0+(n+1)\cdot T_s)), \\ y\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right) &= C_2\cdot S_1\cdot x(t_0+n\cdot T_s) \\ &+C_2\cdot T_1\cdot\sin(\omega\cdot(t_0+n\cdot T_s))+C_2\cdot T_2\cdot\cos(\omega\cdot(t_0+n\cdot T_s)) \\ &-C_2\cdot U_1\cdot\sin\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)-C_2\cdot U_2\cdot\cos\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right) \\ &+D_2\cdot\sin\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right) \ \textbf{and} \\ y(t_0+(n+1)\cdot T_s) &= C_1\cdot S_2\cdot x\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right) \\ &+C_1\cdot T_1'\cdot\sin\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right)+C_1\cdot T_2'\cdot\cos\left(\omega\cdot\left(t_0+\left(n+\frac{1}{2}\right)\cdot T_s\right)\right) \\ &-C_1\cdot U_1'\cdot\sin(\omega\cdot(t_0+(n+1)\cdot T_s))-C_1\cdot U_2'\cdot\cos(\omega\cdot(t_0+(n+1)\cdot T_s)) \\ &+D_1\cdot\sin(\omega\cdot(t_0+(n+1)\cdot T_s)) \qquad \textbf{(27),} \end{aligned}$$

respectively, where

$$S_1=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot e^{\frac{\lambda_i\cdot T_s}{2}}\cdot A_1^{j-1},$$

$$T_1=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A_1^{j-1}\cdot B_1\cdot\frac{e^{\frac{\lambda_i\cdot T_s}{2}}}{\lambda_i\cdot\left(1+\frac{\omega^2}{\lambda_i^2}\right)},$$

$$T_2=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A_1^{\,j-1}\cdot B_1\cdot\frac{\omega\cdot e^{\frac{\lambda_i\cdot T_s}{2}}}{\lambda_i^{\,2}\cdot\left(1+\frac{\omega^2}{\lambda_i^{\,2}}\right)},$$

$$U_1=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A_1^{\,j-1}\cdot B_1\cdot\frac{1}{\lambda_i\cdot\left(1+\frac{\omega^2}{\lambda_i^{\,2}}\right)},$$

$$U_2=\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ji}\cdot A_1^{\,j-1}\cdot B_1\cdot\frac{\omega}{\lambda_i^{\,2}\cdot\left(1+\frac{\omega^2}{\lambda_i^{\,2}}\right)},$$

$$S_2=\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot e^{\frac{\lambda'_i\cdot T_s}{2}}\cdot A_2^{\,j-1},$$

$$T'_1=\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot A_2^{\,j-1}\cdot B_2\cdot\frac{e^{\frac{\lambda'_i\cdot T_s}{2}}}{\lambda'_i\cdot\left(1+\frac{\omega^2}{\lambda_i'^2}\right)},$$

$$T'_2=\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot A_2^{\,j-1}\cdot B_2\cdot\frac{\omega\cdot e^{\frac{\lambda'_i\cdot T_s}{2}}}{\lambda_i'^2\cdot\left(1+\frac{\omega^2}{\lambda_i'^2}\right)},$$

$$U'_1=\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot A_2^{\,j-1}\cdot B_2\cdot\frac{1}{\lambda'_i\cdot\left(1+\frac{\omega^2}{\lambda_i'^2}\right)},$$

$$U'_2=\sum_{j=1}^{N}\sum_{i=1}^{N}a'_{ji}\cdot A_2^{\,j-1}\cdot B_2\cdot\frac{\omega}{\lambda_i'^2\cdot\left(1+\frac{\omega^2}{\lambda_i'^2}\right)} \qquad \textbf{(28).}$$

By solving the difference equation (27), we have:

$$\begin{aligned}
&x(t_0+n\cdot T_s)=(S_2\cdot S_1)^n\cdot x(t_0)\\
&+(S_2\cdot S_1)^{n-1}\cdot S_2\cdot T_1\cdot\sin(\omega\cdot t_0)+(S_2\cdot S_1)^{n-1}\cdot S_2\cdot T_2\cdot\cos(\omega\cdot t_0)\\
&+\sum_{p=1}^{n}(S_2\cdot S_1)^{n-p}\cdot(T'_1-S_2\cdot U_1)\cdot\sin\left(\omega\cdot\left(t_0+\left(p-\frac{1}{2}\right)\cdot T_s\right)\right)\\
&+\sum_{p=1}^{n}(S_2\cdot S_1)^{n-p}\cdot(T'_2-S_2\cdot U_2)\cdot\cos\left(\omega\cdot\left(t_0+\left(p-\frac{1}{2}\right)\cdot T_s\right)\right)\\
&+\sum_{p=1}^{n-1}(S_2\cdot S_1)^{n-p-1}\cdot S_2\cdot(T_1-S_1\cdot U'_1)\cdot\sin(\omega\cdot(t_0+p\cdot T_s))\\
&+\sum_{p=1}^{n-1}(S_2\cdot S_1)^{n-p-1}\cdot S_2\cdot(T_2-S_1\cdot U'_2)\cdot\cos(\omega\cdot(t_0+p\cdot T_s))\\
&-U'_1\cdot\sin(\omega\cdot(t_0+n\cdot T_s))-U'_2\cdot\cos(\omega\cdot(t_0+n\cdot T_s))\ \textbf{and}\\
&y(t_0+n\cdot T_s)=C_1\cdot(S_2\cdot S_1)^n\cdot x(t_0)\\
&+C_1\cdot(S_2\cdot S_1)^{n-1}\cdot S_2\cdot T_1\cdot\sin(\omega\cdot t_0)+C_1\cdot(S_2\cdot S_1)^{n-1}\cdot S_2\cdot T_2\cdot\cos(\omega\cdot t_0)\\
&+\sum_{p=1}^{n}C_1\cdot(S_2\cdot S_1)^{n-p}\cdot(T'_1-S_2\cdot U_1)\cdot\sin\left(\omega\cdot\left(t_0+\left(p-\frac{1}{2}\right)\cdot T_s\right)\right)\\
&+\sum_{p=1}^{n}C_1\cdot(S_2\cdot S_1)^{n-p}\cdot(T'_2-S_2\cdot U_2)\cdot\cos\left(\omega\cdot\left(t_0+\left(p-\frac{1}{2}\right)\cdot T_s\right)\right)\\
&+\sum_{p=1}^{n-1}C_1\cdot(S_2\cdot S_1)^{n-p-1}\cdot S_2\cdot(T_1-S_1\cdot U'_1)\cdot\sin(\omega\cdot(t_0+p\cdot T_s))\\
&+\sum_{p=1}^{n-1}C_1\cdot(S_2\cdot S_1)^{n-p-1}\cdot S_2\cdot(T_2-S_1\cdot U'_2)\cdot\cos(\omega\cdot(t_0+p\cdot T_s))\\
&-C_1\cdot U'_1\cdot\sin(\omega\cdot(t_0+n\cdot T_s))-C_1\cdot U'_2\cdot\cos(\omega\cdot(t_0+n\cdot T_s))\\
&+D_1\cdot\sin(\omega\cdot(t_0+n\cdot T_s)) \qquad \textbf{(29),}
\end{aligned}$$

respectively.

Error equations

From equations (24) and (29), the prediction errors are:

$$\begin{aligned}
&\Delta x(t_0+n\cdot T_s)=G_0(n)\cdot x(t_0)+\sum_{p=0}^{2\cdot n}L_p(n)\cdot\sin\left(\omega\cdot\left(t_0+\frac{p}{2}\cdot T_s\right)\right)\\
&\quad+\sum_{p=0}^{2\cdot n}M_p(n)\cdot\cos\left(\omega\cdot\left(t_0+\frac{p}{2}\cdot T_s\right)\right)\ \textbf{and}\\
&\Delta y(t_0+n\cdot T_s)=G'_0(n)\cdot x(t_0)+\sum_{p=0}^{2\cdot n}L'_p(n)\cdot\sin\left(\omega\cdot\left(t_0+\frac{p}{2}\cdot T_s\right)\right)\\
&+\sum_{p=0}^{2\cdot n}M'_p(n)\cdot\cos\left(\omega\cdot\left(t_0+\frac{p}{2}\cdot T_s\right)\right) \qquad \textbf{(30),}
\end{aligned}$$

respectively. If $G_0(n)$ and $G'_0(n)$ converge to a constant, then the error at the steady state can be approximately by the sinusoids with a DC offset. That is:

$$\begin{aligned}
&\Delta x(t_0+n\cdot T_s)=\sum_{p=0}^{2\cdot n}\Psi_p(n)\cdot\sin\left(\omega\cdot\left(t_0+\frac{p}{2}\cdot T_s\right)+\phi_p\right)+\Omega\ \textbf{and}\\
&\Delta y(t_0+n\cdot T_s)=\sum_{p=0}^{2\cdot n}\Psi'_p(n)\cdot\sin\left(\omega\cdot\left(t_0+\frac{p}{2}\cdot T_s\right)+\phi'_p\right)+\Omega' \quad \textbf{(31).}
\end{aligned}$$

From equation (31), the error is mainly contributed by the phase shift and the amplitude of the sinusoids, which may be significant.

VI. SIMULATION RESULTS

Figure 1 shows a schematic example of a switching circuit [5]. Figure 2 and figure 3 show the output of the circuit predicted by the state-space averaging method and the state-space piecewise method for different polynomial inputs, respectively. Figure 4 and figure 5 show the corresponding prediction error and the percentage error.

Figure 6 and figure 7 show the output predicted by the state-space averaging method and the state-space piecewise method for a sinusoidal input, respectively. Figure 8 and figure 9 show the corresponding prediction error and the percentage error.

VII. CONCLUDING REMARKS

The validity of the state-space averaging method is studied in this paper. It is found that, in general, the state-space averaging approach is applicable as a useful tool for the analysis and design of switching circuits with polynomial input, e.g., DC-DC converter. However, it may not provide an appropriate model for the analysis and design for non-polynomial input systems, e.g., AC-AC converter, as the prediction error terms are to be quite large.

ACKNOWLEDGEMENT

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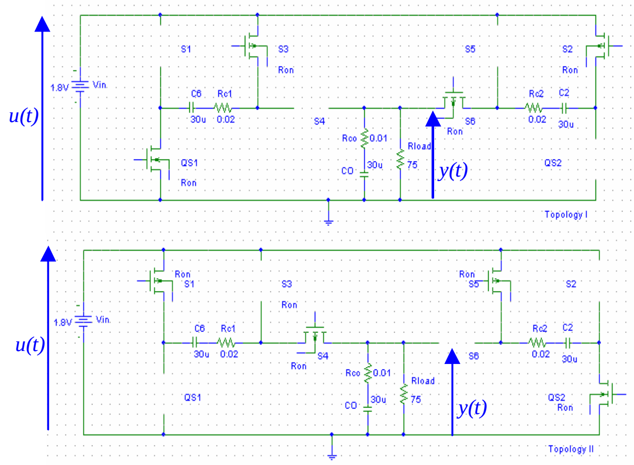

Fig. 1. Schematic example of a switching circuit

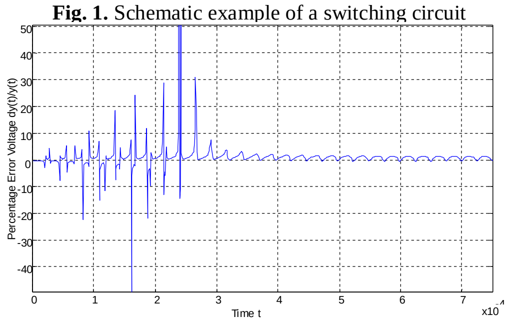

Fig. 9. Percentage Error Voltage for Sinusoidal Input

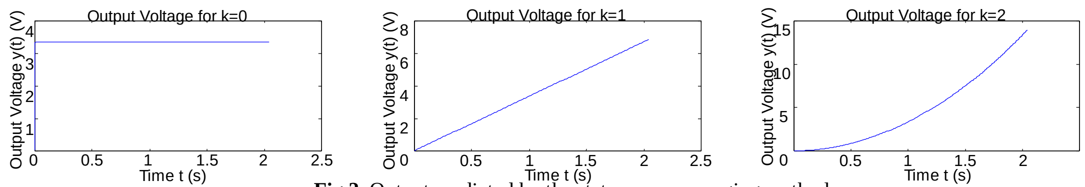

Fig 2. Output predicted by the state-space averaging method

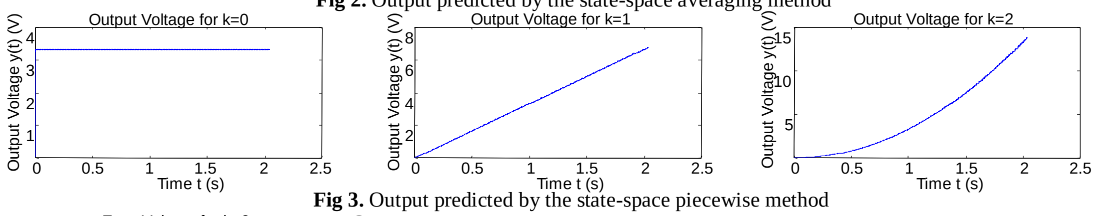

Fig 3. Output predicted by the state-space piecewise method

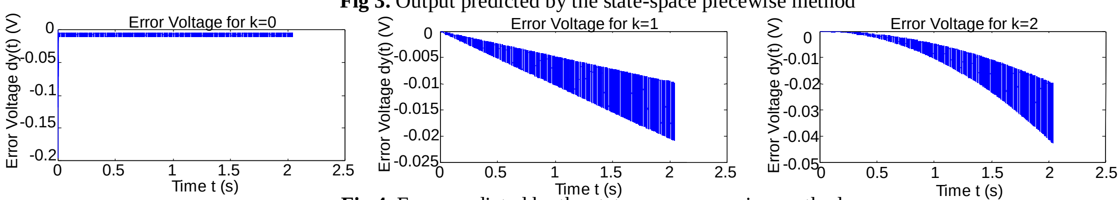

Fig 4. Error predicted by the state-space averaging method

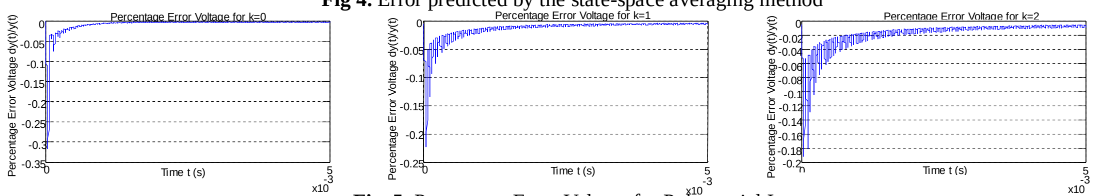

Fig. 5. Percentage Error Voltage for Polynomial Inputs

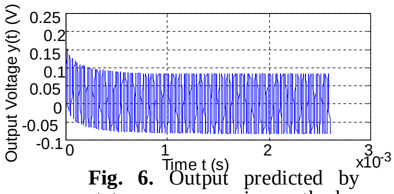

Fig. 6. Output predicted by state-space averaging method

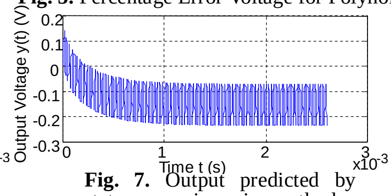

Fig. 7. Output predicted by state-space piecewise method

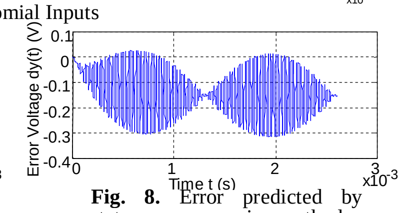

Fig. 8. Error predicted by state-space averaging method