

Representation of Perfect Reconstruction Octave Decomposition Filter Banks with Set of Decimators $\{2,4,4\}$ via Tree Structure

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Abstract—In this letter, we prove that a filter bank with set of decimators $\{2,4,4\}$ achieves perfect reconstruction if and only if it can be represented via a tree structure and each branch of the tree structure achieves perfect reconstruction.

Index Terms—filter bank, perfect reconstruction, tree structure.

I. INTRODUCTION

IT is well known that the tree structure filter bank shown in figure 1b can be represented via an octave decomposition filter bank with set of decimators $\{2,4,4\}$ shown in figure 1a, and perfect reconstruction can be achieved if and only if each branch of the tree structure achieves perfect reconstruction [1]-[4]. However, is the converse true? That is, given any perfect reconstruction octave decomposition filter bank with set of decimators $\{2,4,4\}$ shown in figure 1a, can it be represented via a tree structure shown in figure 1b? In general, a perfect reconstruction non-uniform filter bank cannot be represented by a tree structure [11], [12]. This letter works on the problem of representing a perfect reconstruction octave decomposition filter bank with set of decimators $\{2,4,4\}$ by a tree structure.

There are some advantages of representing an octave decomposition filter bank with set of decimators $\{2,4,4\}$ via a tree structure, such as reducing the filter length in the filters [5], and improving the computation complexity and implementation speed [5]. In section II, we show how a perfect reconstruction octave decomposition filter bank with set of decimators $\{2,4,4\}$ can be converted to a tree structure filter bank. Some illustrative examples are demonstrated in section III. Finally, a conclusion is given in section IV.

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II. REPRESENTATION OF OCTAVE DECOMPOSITION FILTER BANK WITH SET OF DECIMATORS $\{2,4,4\}$ VIA A TREE STRUCTURE

Theorem 1: An octave decomposition filter bank with set of decimators $\{2,4,4\}$ achieves perfect reconstruction if and only if it can be represented via a tree structure and each branch of the tree structure achieves perfect reconstruction.

Proof: Since the if part is well known [1]-[4], we only prove the only if part.

An octave decomposition filter bank with set of decimators $\{2,4,4\}$ shown in figure 1a achieves perfect reconstruction if and only if $\exists c \in C$ and $\exists m \in Z$ such that:

$$\begin{bmatrix} \frac{1}{2} \cdot H_0(z) & \frac{1}{4} \cdot H_1(z) & \frac{1}{4} \cdot H_2(z) \\ 0 & \frac{1}{4} \cdot H_1(z \cdot W) & \frac{1}{4} \cdot H_2(z \cdot W) \\ \frac{1}{2} \cdot H_0(z \cdot W^2) & \frac{1}{4} \cdot H_1(z \cdot W^2) & \frac{1}{4} \cdot H_2(z \cdot W^2) \\ 0 & \frac{1}{4} \cdot H_1(z \cdot W^3) & \frac{1}{4} \cdot H_2(z \cdot W^3) \end{bmatrix} \cdot \begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} c \cdot z^m \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (1)$$

where

$$W = \exp\left(-\frac{j \cdot 2 \cdot \pi}{4}\right) = -j. \quad (2)$$

This directly implies that:

$$\det\left(\begin{bmatrix} H_1(z \cdot W) & H_2(z \cdot W) \\ H_1(z \cdot W^3) & H_2(z \cdot W^3) \end{bmatrix}\right) = 0. \quad (3)$$

Since (3) is true for all $z \in C$, by using the substitution $z = z \cdot W^{-1}$, we have:

$$\det\left(\begin{bmatrix} H_1(z) & H_2(z) \\ H_1(z \cdot W^2) & H_2(z \cdot W^2) \end{bmatrix}\right) = H_1(z) \cdot H_2(z \cdot W^2) - H_2(z) \cdot H_1(z \cdot W^2) = 0. \quad (4)$$

$$\Rightarrow \frac{H_1(z)}{H_2(z)} = \frac{H_1(z \cdot W^2)}{H_2(z \cdot W^2)} = R(z) \quad (5)$$

Let

$$H_i(z) = \sum_{l=0}^3 z^{-l} \cdot E_{i,l}(z^4), \quad (6)$$

for $i = 1, 2$,

then:

$$\begin{aligned} & \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4) + z^{-1} \cdot (E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4))}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4) + z^{-1} \cdot (E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4))} \\ &= \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4) - z^{-1} \cdot (E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4))}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4) - z^{-1} \cdot (E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4))} \quad (7) \\ &= R(z). \\ &\Rightarrow \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4) + z^{-1} \cdot (E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4))}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4) + z^{-1} \cdot (E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4))} \\ &= \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4)}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4)} \quad (8) \\ &= \frac{E_{1,1}(z^4) + z^{-2} \cdot E_{1,3}(z^4)}{E_{2,1}(z^4) + z^{-2} \cdot E_{2,3}(z^4)} \\ &= R(z). \end{aligned}$$

Hence:

$$\frac{H_1(z)}{H_2(z)} = \frac{E_{1,0}(z^4) + z^{-2} \cdot E_{1,2}(z^4)}{E_{2,0}(z^4) + z^{-2} \cdot E_{2,2}(z^4)} = R(z). \quad (9)$$

As a result, there exist $F_1'(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z) = F_1'(z) \cdot F_0(z^2)$ and $H_2(z) = F_1'(z) \cdot F_1(z^2)$, respectively. Hence, the octave decomposition filter bank with set of decimators $\{2,4,4\}$ shown in figure 1a can be represented via a tree structure shown in figure 1b.

Because the input-output behavior of these two representations remains unchanged, so if the octave decomposition filter bank with set of decimators $\{2,4,4\}$ achieves perfect reconstruction, the tree structure filter bank does also. Hence, it completes the proof. ■

For the design of the synthesis filters in an octave decomposition filter bank with set of decimators $\{2,4,4\}$, if that filter bank system achieves perfect reconstruction, it is proved in Theorem 1 that it can be represented by a tree-structure filter bank and that tree-structure filter bank achieves perfect reconstruction. The synthesis filters in a tree-structure filter bank for the perfect reconstruction can be designed easily [7], and those filters in the tree-structure filter bank can be transformed to the filters in an octave decomposition filter bank with set of decimators $\{2,4,4\}$. Hence, the synthesis filters in an octave decomposition filter bank with set of decimators $\{2,4,4\}$ can be designed.

III. ILLUSTRATIVE EXAMPLES

A. Non-tree Structure Filter Bank

Consider an example of $H_0(z) = 1 + z^{-1}$, $H_1(z) = 1 - z^{-1}$, and $H_2(z) = z^{-2}$, respectively. Since $H_1(z)/H_2(z) = z^2 \cdot (1 - z^{-1})$, there do not exist $F_1'(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z) = F_1'(z) \cdot F_0(z^2)$ and $H_2(z) = F_1'(z) \cdot F_1(z^2)$, respectively. Hence, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ cannot be represented via a tree structure. By Theorem 1, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ does not achieve perfect reconstruction.

It is worth to note that by converting the octave decomposition filter bank with set of decimators $\{2,4,4\}$ to a uniform filter bank shown in figure 2 [6]-[10], perfect reconstruction can be achieved. However, $G_{-1}'(z) \neq z^{-2} \cdot G_0'(z)$. This implies that the corresponding synthesis filter $G_0(z)$ shown in figure 1a is time varying.

B. Tree Structure Filter Bank

Consider another example with $H_0(z) = 1 + z^{-1}$, $H_1(z) = 1 - z^{-1} + z^{-2} - z^{-3}$, and $H_2(z) = 1 - z^{-1} - z^{-2} + z^{-3}$, respectively. Since $H_1(z)/H_2(z) = (1 + z^{-2})/(1 - z^{-2})$, there exist $F_1'(z)$, $F_0(z)$, and $F_1(z)$ such that $H_1(z) = F_1'(z) \cdot F_0(z^2)$ and $H_2(z) = F_1'(z) \cdot F_1(z^2)$, respectively. Hence, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ can be represented via a tree structure. It can be checked easily that each branch in the tree structure can achieve perfect reconstruction. Hence, this octave decomposition filter bank with set of decimators $\{2,4,4\}$ can achieve perfect reconstruction.

IV. CONCLUSION

In this letter, we show that an octave decomposition filter bank with set of decimators $\{2,4,4\}$ achieves perfect reconstruction if and only if it can be represented via a tree structure and each branch of the tree structure achieves perfect reconstruction. The advantage of representing an octave decomposition filter bank with set of decimators $\{2,4,4\}$ via a tree structure is to reduce the computation complexity and provide a fast implementation for an octave decomposition filter bank with set of decimators $\{2,4,4\}$ [5].

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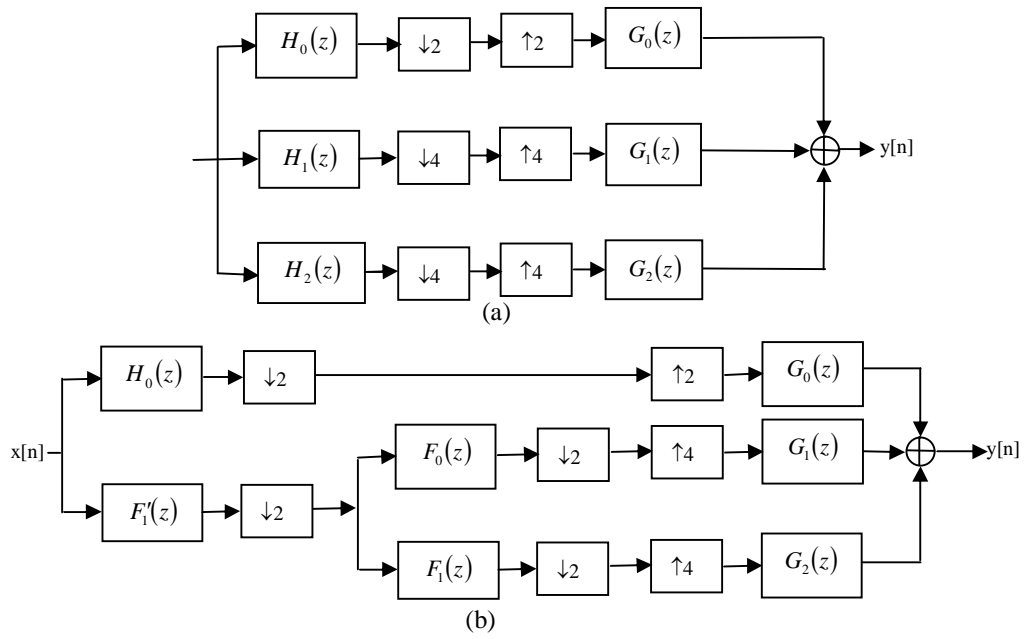


Fig. 1. (a) Octave decomposition filter bank with set of decimators $\{2,4,4\}$.
 (b) tree structure filter bank.

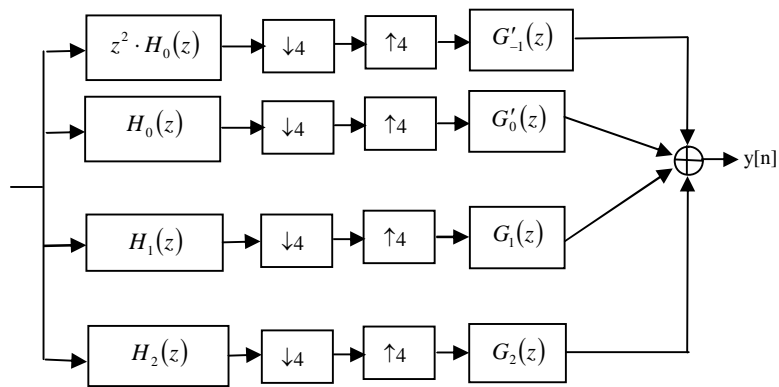


Fig. 2. Representation of octave decomposition filter bank with set of decimators $\{2,4,4\}$ via a uniform filter bank.