New Results on Periodic Symbolic Sequences of Second Order Digital Filters with Two's Complement Arithmetic

Bingo Wing-Kuen Ling and Peter Kwong-Shun Tam

Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China

SUMMARY

In this article, the second order digital filter with two's complement arithmetic in [1] is considered. Necessary conditions for the symbolic sequences to be periodic after a number of iterations are given when the filter parameters are at $b=a+1$ and $b=-a+1$. Furthermore, for some particular values of a, even when one of the eigenvalues is outside the unit circle, the system may behave as a linear system after a number of iterations and the state vector may toggle between two states or converge to a fixed point at the steady state. The necessary and sufficient conditions for these phenomena are given in this article.

KEY WORDS: second order digital filter; two's complement arithmetic; symbolic sequences; eigenvalues

1. INTRODUCTION

A nonlinear behavior may occur on a second order digital filter when the filter is implemented using a two's complement arithmetic for the addition operation. To analyze such a behavior, a symbolic analysis was proposed and the admissibility of the symbolic sequences was studied for the special case when b=-1 and $|a|$<2 [1-4, 7]. However, even when the symbolic sequence is admissible, there are many possibilities. In order to study the various possibilities, the set of admissible sequences can be partitioned into three subsets: One set contains periodic symbolic sequences. The second set contains symbolic sequences that are periodic after a number of iterations. The third set contains symbolic sequences that are never periodic. Some results on the first two sets were obtained for b=-1 and $|a|$<2 [1-4, 7]. These results have been extended for other real values of a, while the value of b is still equal to –1 [5].

However, will those existing results remain valid if the filter parameter $b \neq -1$? Specifically, we are interested in the nonlinear behavior which may occur when $b = a + 1$ and $b = -a + 1$, and the answer to the following questions: Under what conditions will the symbolic sequence be periodic? If the symbolic sequence is periodic, under what conditions will the system behave as a linear system after a number of iterations and the state vector toggles among several states or converges to a fixed point? In this article, we focus on both the cases of $b = a + 1$ and $b = -a + 1$.

In section 2, we will present the notations used in the existing literatures [1-7], and this article will also employ the same set of notations. In section 3, some new results on the above problems are presented. Finally, a conclusion is summarized in section 4.

2. NOTATIONS

The notations used in [1-7] are adopted as follows:

The system is defined as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathrm{F}\left(\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}\right) = \begin{bmatrix} x_2(k) \\ f(b \cdot x_1(k) + a \cdot x_2(k)) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k \tag{1}$$

where $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in I^2 \equiv \{(x_1, x_2) : -1 \le x_1 < 1, -1 \le x_2 < 1\}$ (2)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{4}$$

$s_k \in \{-m, \cdots, -1, 0, 1, \cdots m\}$ where m is the minimum integer satisfying

$$-2 \cdot m - 1 \le b \cdot x_1 + a \cdot x_2 < 2 \cdot m + 1 \tag{5}$$

and $f(x) = x - 2 \cdot n$ such that $2 \cdot n - 1 \le x < 2 \cdot n + 1$ and $n \in \mathrm{Z}^+ \cup \{0\}$ (6)

Given an initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in I^2$, a symbolic sequence $s = (s_0, s_1, \cdots) \in \Sigma$ can be generated by the map $\mathrm{S} : I^2 \to \Sigma$, and a sequence s in Σ is admissible if $\exists \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in I^2$ such that $\mathrm{S}\left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\right) = s$.

The set Σ can be partitioned into three subsets: $\Sigma_\alpha = \{s = (s_0 s_1 s_2 \cdots) : s \text{ is periodic}\}$, $\Sigma_\beta = \{s = (s_0 s_1 s_2 \cdots) : s \text{ is periodic after a number of iterations}\}$ and $\Sigma_\gamma = \Sigma \setminus (\Sigma_\alpha \cup \Sigma_\beta)$.

3. NEW RESULTS FOR THE PERIODIC SYMBOLIC SEQUENCES

This section presents several conditions for the state vector to be periodic after a number of iterations. These conditions can be stated in the following lemmas, theorems and remarks, where the stability of these periodic orbits is stated in the observations:

Lemma 1

For $b = a+1$, *if* $\exists M \in Z^{+}$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+M) \\ x_2(k+M) \end{bmatrix}$ for $k \geq k_0$, *then*

$$\left(x_1(0)+x_2(0)\right)\cdot\left((a+1)^M - 1\right) + 2\cdot\left(2\cdot(a+1)^M - 1\right)\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}} + 2\cdot(a+1)^M\cdot\sum_{j=k_0}^{k_0+M-1}\frac{s_j}{(a+1)^{j+1}} = 0\,.$$

Proof

For $b = a+1$, the nonlinear system in (1) can be represented by the state equation $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}\cdot\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cdot s_k$. Hence, the solution of the system is:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a+2}\cdot\begin{bmatrix} (a+1)^k + (a+1)\cdot(-1)^k & (a+1)^k - (-1)^k \\ (a+1)^{k+1} - (a+1)\cdot(-1)^k & (a+1)^{k+1} + (-1)^k \end{bmatrix}\cdot\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \frac{2}{a+2}\cdot\sum_{j=0}^{k-1}\begin{bmatrix} (a+1)^{k-1-j} - (-1)^{k-1-j} \\ (a+1)^{k-j} + (-1)^{k-1-j} \end{bmatrix}\cdot s_j \quad (7)$$

If $\exists M \in Z^{+}$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+M) \\ x_2(k+M) \end{bmatrix}$ for $k \geq k_0$, then:

$$\begin{aligned} &\frac{1}{a+2}\cdot\begin{bmatrix} (a+1)^{k_0} + (a+1)\cdot(-1)^{k_0} & (a+1)^{k_0} - (-1)^{k_0} \\ (a+1)^{k_0+1} - (a+1)\cdot(-1)^{k_0} & (a+1)^{k_0+1} + (-1)^{k_0} \end{bmatrix}\cdot\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \frac{2}{a+2}\cdot\sum_{j=0}^{k_0-1}\begin{bmatrix} (a+1)^{k_0-1-j} - (-1)^{k_0-1-j} \\ (a+1)^{k_0-j} + (-1)^{k_0-1-j} \end{bmatrix}\cdot s_j \\ &= \frac{1}{a+2}\cdot\begin{bmatrix} (a+1)^{k_0+M} + (a+1)\cdot(-1)^{k_0+M} & (a+1)^{k_0+M} - (-1)^{k_0+M} \\ (a+1)^{k_0+M+1} - (a+1)\cdot(-1)^{k_0+M} & (a+1)^{k_0+M+1} + (-1)^{k_0+M} \end{bmatrix}\cdot\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1}\begin{bmatrix} (a+1)^{k_0+M-1-j} - (-1)^{k_0+M-1-j} \\ (a+1)^{k_0+M-j} + (-1)^{k_0+M-1-j} \end{bmatrix}\cdot s_j \end{aligned} \quad (8)$$

Hence,

$$\begin{cases} \left((a+1)^M - 1\right)\cdot\frac{(a+1)^{k_0}}{a+2}\cdot\left(x_1(0)+x_2(0)+2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}\right) + \left((-1)^M - 1\right)\cdot\frac{(-1)^{k_0}}{a+2}\cdot\left((a+1)\cdot x_1(0) - x_2(0) - 2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(-1)^{j+1}}\right) \\ + \frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1} s_j\cdot\left((a+1)^{k_0+M-1-j} - (-1)^{k_0+M-1-j}\right) = 0 \end{cases} \quad (9)$$

$$\begin{cases} \left((a+1)^M - 1\right)\cdot\frac{(a+1)^{k_0+1}}{a+2}\cdot\left(x_1(0)+x_2(0)+2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}\right) + \left((-1)^M - 1\right)\cdot\frac{(-1)^{k_0}}{a+2}\cdot\left(2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(-1)^{j+1}} - \left((a+1)\cdot x_1(0) - x_2(0)\right)\right) \\ + \frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1} s_j\cdot\left((a+1)^{k_0+M-j} + (-1)^{k_0+M-1-j}\right) = 0 \end{cases} \quad (10)$$

Let

$$t_1 = \left((a+1)^M - 1\right) \cdot \frac{(a+1)^{k_0}}{a+2} \cdot \left(x_1(0) + x_2(0) + 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} \right) \tag{11}$$

$$t_2 = \left((-1)^M - 1\right) \cdot \frac{(-1)^{k_0}}{a+2} \cdot \left((a+1) \cdot x_1(0) - x_2(0) - 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(-1)^{j+1}} \right) \tag{12}$$

$$t_3 = \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot (a+1)^{k_0+M-1-j} \tag{13}$$

$$t_4 = \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot (-1)^{k_0+M-1-j} \tag{14}$$

Then we have:

$$\begin{cases} t_1 + t_2 + t_3 - t_4 = 0 \\ (a+1) \cdot t_1 - t_2 + (a+1) \cdot t_3 + t_4 = 0 \end{cases} \tag{15}$$

which

$$\Rightarrow a = -2 \quad \text{or} \quad t_1 = -t_3 \tag{16}$$

For $a \neq -2$, we have $t_1 = -t_3$, that is:

$$\left((a+1)^M - 1\right) \cdot \frac{(a+1)^{k_0}}{a+2} \cdot \left(x_1(0) + x_2(0) + 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} \right) = -\frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot (a+1)^{k_0+M-1-j} \tag{17}$$

$$\Rightarrow (x_1(0) + x_2(0)) \cdot \left((a+1)^M - 1\right) + 2 \cdot \left(2 \cdot (a+1)^M - 1\right) \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} + 2 \cdot (a+1)^M \cdot \sum_{j=k_0}^{k_0+M-1} \frac{s_j}{(a+1)^{j+1}} = 0 \quad \blacksquare$$

Remark 1

If, after a number of iterations, the state vector is periodic with period *M*, then the symbolic sequence will also be periodic with the same period, that is, $s \in \Sigma_\beta$. Hence, this lemma gives a necessary condition for a symbolic sequence to be periodic after a number of iterations.

However, s_k is an integer in $\{-m, \cdots, -1, 0, 1, \cdots m\}$ and the periodicity of the symbolic sequence is *M*. So there are $(2 \cdot m + 1)^M$ possibilities. In the following theorem, the case of $s_k = 0$ for $k \ge k_0$ is discussed.

Theorem 1

For $b = a+1$ and $|a+1| > 1$, $s_k = 0$ for $k \ge k_0$ *if and only if* $\exists k_0 \in Z^+ \bigcup \{0\}$ such that $x_1(k_0) = -x_2(k_0)$.

Proof

For the *if* part, since $b = a+1$ and $s_k = 0$ for $k \ge k_0$, we have:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a+2} \cdot \begin{bmatrix} (a+1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0)) + (-1)^{k-k_0} \cdot ((a+1) \cdot x_1(k_0) - x_2(k_0)) \\ (a+1)^{k-k_0+1} \cdot (x_1(k_0) + x_2(k_0)) - (-1)^{k-k_0} \cdot ((a+1) \cdot x_1(k_0) - x_2(k_0)) \end{bmatrix} \quad (18)$$

Since $|a+1| > 1$, $(a+1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0))$ diverges as $k \to +\infty$ if $x_1(k_0) + x_2(k_0) \ne 0$. However, as $s_k = 0$ for $k \ge k_0$, this implies that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in I^2$ and that $(a+1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0))$ is bounded for $k \ge k_0$. Hence, $x_1(k_0) + x_2(k_0) = 0$ and this proves the *if* part.

For the *only if* part, since $x_1(k_0) = -x_2(k_0)$, we have:

$\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ f((a+1) \cdot x_1(k_0) - a \cdot x_1(k_0)) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ f(x_1(k_0)) \end{bmatrix}$. Since $|x_1(k_0)| < 1$, we have $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ x_1(k_0) \end{bmatrix}$ and $s_{k_0} = 0$.

Similarly, we have $\begin{bmatrix} x_1(k_0+2) \\ x_2(k_0+2) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ -x_1(k_0) \end{bmatrix}$ and $s_{k_0+1} = 0$. Since $\begin{bmatrix} x_1(k_0+2) \\ x_2(k_0+2) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix}$, we have $s_k = 0$ for $k \ge k_0$ and this proves the *only if* part. ■

Remark 2

The eigenvalues of matrix **A** is -1 and $a+1$. Since $|a+1| > 1$, one of the eigenvalues is outside the unit circle. However, the system may behave as a linear system when $s_k = 0$ for $k \geq k_0$. Theorem 1 gives the necessary and sufficient condition for the nonlinear system to behave as a linear system after a number of iterations. It is interesting to note that the system may behave as a linear system after a number of iterations if and only if the state vector toggles between two points on a particular straight line of the phase portrait.

Example 1

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.9003 \\ -0.5377 \end{bmatrix}$. Figure 1 shows the phase portrait of the system. It can be seen from the figure that the state vector toggles between two states at the steady state on a particular straight line $x_1 = -x_2$ of the phase portrait.

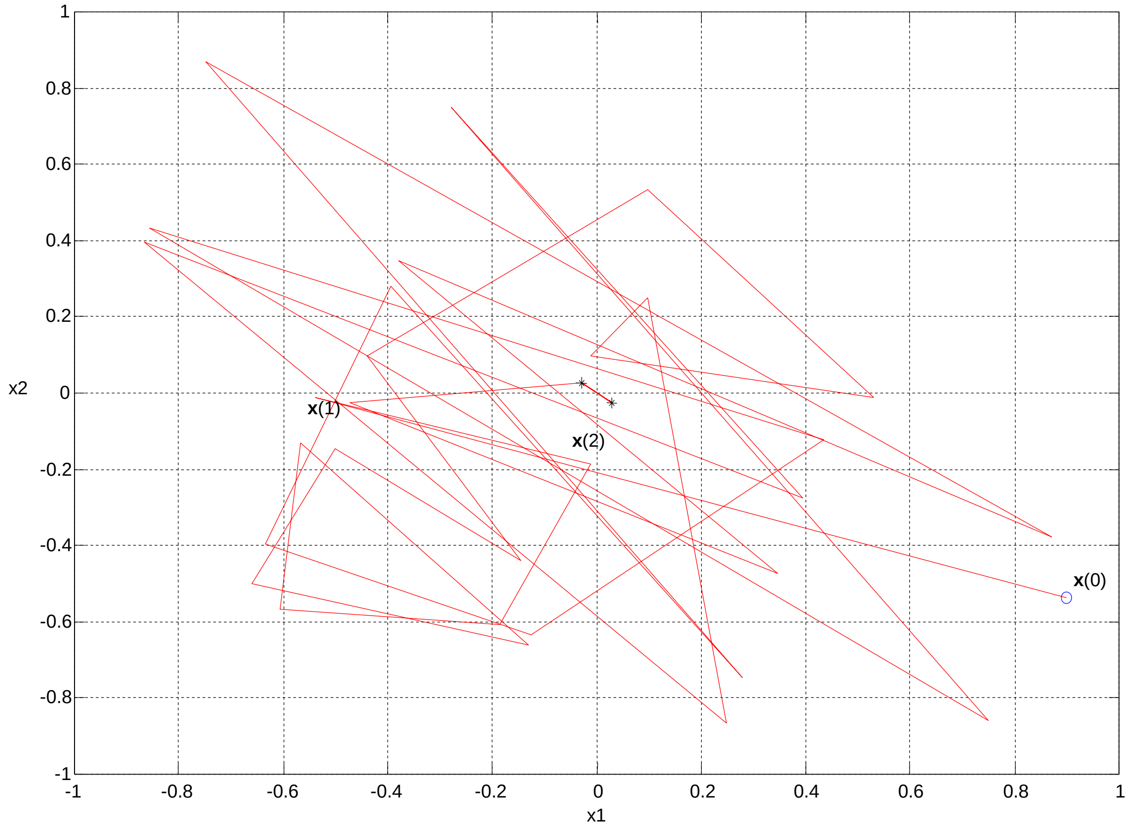

Figure 1. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the points with '*' denote the 'steady states' of **x**.

We have discussed the case when $s_k = 0$ for $k \ge k_0$ in theorem 1. What happens when $s_k \ne 0$ for $k \ge k_0$? We will present an interesting result in the following theorem:

Theorem 2

For $b = a + 1$ and a being an odd integer, $\exists k_0 \in Z^+ \cup \{0\}$ such that $x_1(k_0) = x_2(k_0) = -1$ *if and only if* $s_k = a$ and $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix}$ for $k \ge k_0$.

Proof

For the *if* part, since $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a+1)\cdot x_1(k_0) + a \cdot x_2(k_0)) \end{bmatrix}$, if $\exists k_0 \in Z^+ \cup \{0\}$ such that $x_1(k_0) = x_2(k_0) = -1$, then we have $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -1 \\ f(-(2\cdot a+1)) \end{bmatrix}$. Since a is an odd integer, we have $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $s_{k_0} = a$. As $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, we have $s_k = a$ and $x_1(k) = x_2(k) = -1$ for $k \ge k_0$.

For the *only if* part, since $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a+1)\cdot x_1(k_0) + a \cdot x_2(k_0)) \end{bmatrix}$, if $x_1(k) = x_1(k_0)$ and $x_2(k) = x_2(k_0)$ for $k \ge k_0$, then we have $\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a+1)\cdot x_1(k_0) + a \cdot x_2(k_0)) \end{bmatrix}$, which implies $x_1(k_0) = x_2(k_0)$ and $x_2(k_0) = f((a+1)\cdot x_1(k_0) + a \cdot x_2(k_0))$. Since $s_k = a$ for $k \ge k_0$, we have $x_1(k_0) = (a+1)\cdot x_1(k_0) + a \cdot x_1(k_0) + 2\cdot a$, which implies $2\cdot a \cdot (x_1(k_0) + 1) = 0$. As a is an odd integer, so $a \ne 0$. As a result, we have $x_1(k_0) = -1$ and we prove the *only if* part.■

Remark 3

Theorem 2 states the necessary and sufficient condition for the state vector to stay at a fixed point after a number of iterations when the parameter a is an odd integer. It is interesting to note that this fixed point is $(-1,-1)$.

Example 2

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$,

$a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.1875 \\ -0.25 \end{bmatrix}$. Figure 2 shows the phase portrait of the system. It can be seen from the figure that the state vector converges to a fixed point $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

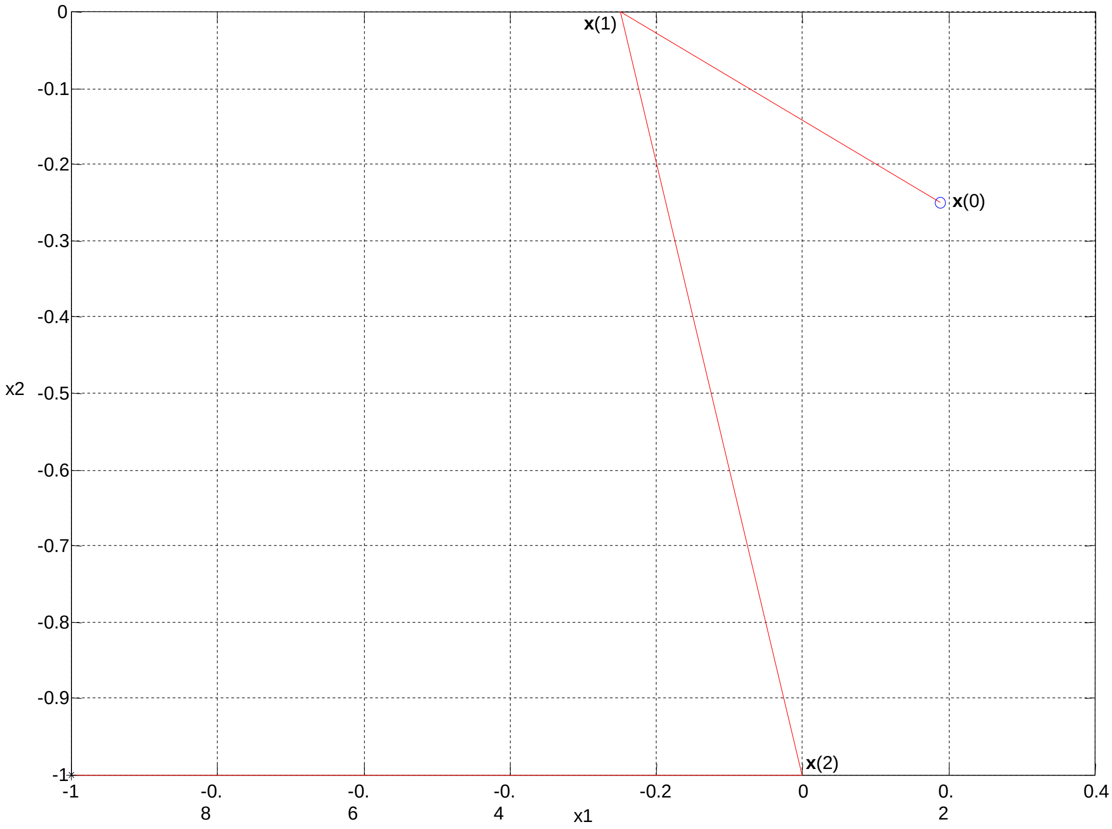

Figure 2. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the point with '*' denotes the 'steady state' of **x**.

Remark 4

The above results are obtained when $b = a+1$. What happens when $b = -a+1$? We can show that results similar to lemma 1 and theorem 1 can be obtained as lemma 2 and theorem 3, while the results of theorem 2 can be modified to that of theorem 4.

Remark 5

Since one of the eigenvalues is unstable, one may predict that the periodic orbits are unstable. However, a counter-intuitive result is found that the periodic orbits are stable if *a* is an odd integer. The result is stated in observation 1 below:

Observation 1

When $b = a + 1$ and *a* is an odd integer, the state vector toggles between two states at the steady state on a particular straight line $x_1 = -x_2$ of the phase portrait or converges to a fixed point $\mathbf{x}^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ for all the initial conditions in I^2.

To demonstrate this phenomenon, a random initial condition $\mathbf{x}(0)$ is generated in I^2, it can be shown in figure 3a that the state converges to a period-*2* signal.

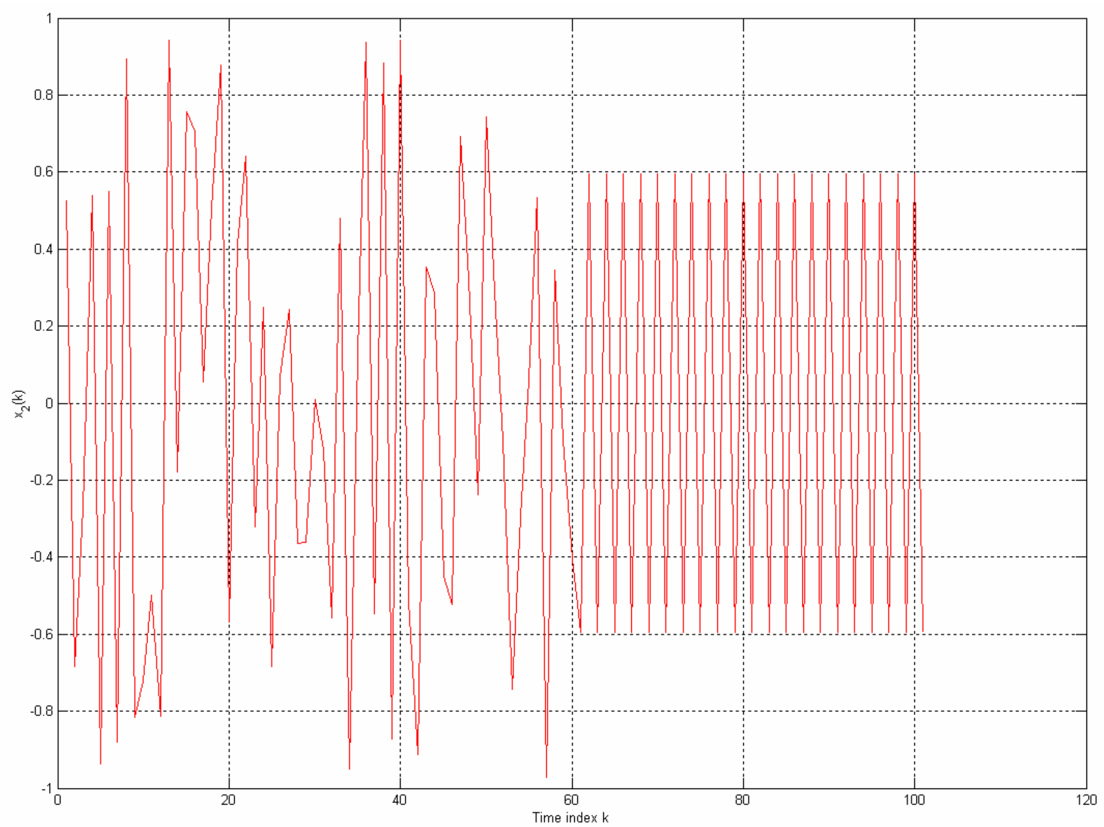

Figure 3a. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition $\mathbf{x}(0)=[0.7826\ 0.5242]^T$ is generated randomly. When *a*=5 and *b*=6, the state converges to a period-*2* signal.

However, when a deviates from an odd integer a little bit, the state neither converges to a periodic signal nor a fixed point, as shown in figure 3b.

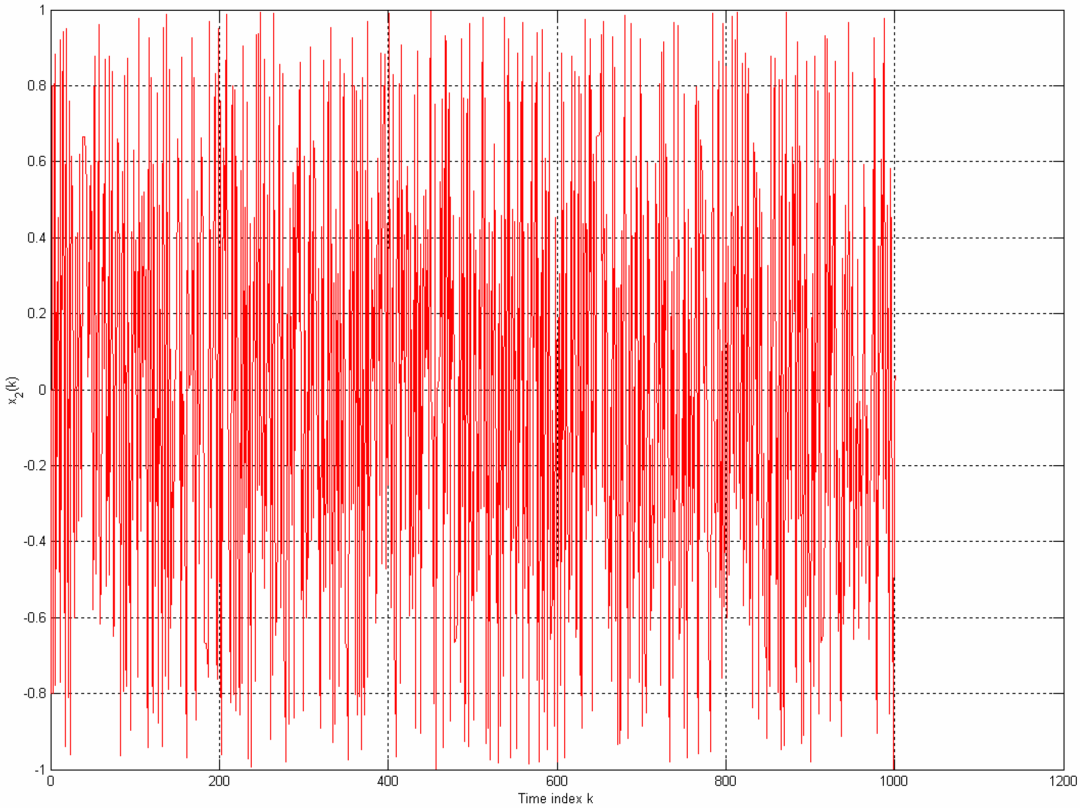

Figure 3b. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[0.8\ -0.7999]^T$, a=3.001 and b=4.001, the state neither converges to a periodic signal nor a fixed point.

<u>Lemma 2</u>

For $b=-a+1$, *if* $\exists M \in Z^+$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+M) \\ x_2(k+M) \end{bmatrix}$ for $k \geq k_0$, *then*

$$\left(1-(a-1)^M\right)\cdot\left(x_1(0)-x_2(0)-2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a-1)^{j+1}}\right)+2\cdot\sum_{j=k_0}^{k_0+M-1}(a+1)^{M-j-1}\cdot s_j=0.$$

<u>Theorem 3</u>

For $b=-a+1$ and $|a-1|>1$, $s_k=0$ for $k \geq k_0$ *if and only if* $\exists k_0 \in Z^+ \cup \{0\}$ such that $x_1(k_0)=x_2(k_0)$.

<u>Example 3</u>

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A}\cdot\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B}\cdot s_k$, where $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -a+1 & a \end{bmatrix}$,

$\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -0.1875 \\ -0.1234 \end{bmatrix}$. Figure 4 shows the phase portrait of the system. It can be seen from the figure that the state vector converges to a fixed point on a particular straight line $x_1 = x_2$ of the phase portrait.

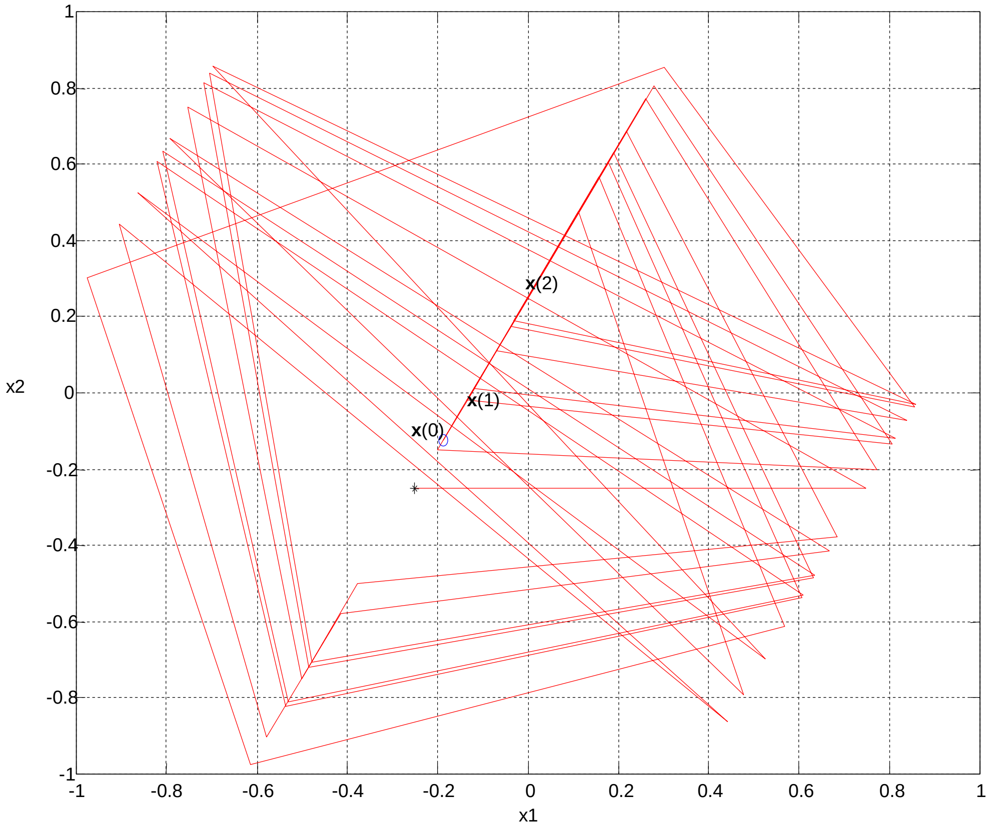

Figure 4. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the point with '*' denotes the 'steady state' of **x**.

Theorem 4

For $b = -a + 1$ and a being an odd integer, there does not exist $k_0 \in Z^+$ such that $s_k = a$ for $k \geq k_0$.

Proof

Since $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \mathbf{A}^{k-k_0} \cdot \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} + \sum_{j=k_0}^{k-1} \mathbf{A}^{k-1-j} \cdot \mathbf{B} \cdot s_j$ for $k > k_0$, if $s_k = a$ for $k \geq k_0$, we have:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a-2} \cdot \begin{bmatrix} (a-1)^{k-k_0} \cdot \left(x_2(k_0) - x_1(k_0) - \frac{2 \cdot a}{2-a} \right) + (a-1) \cdot x_1(k_0) - x_2(k_0) + \frac{2 \cdot a}{2-a} - 2 \cdot a \cdot (k - k_0) \\ (a-1)^{k-k_0+1} \cdot \left(x_2(k_0) - x_1(k_0) - \frac{2 \cdot a}{2-a} \right) + (a-1) \cdot x_1(k_0) - x_2(k_0) + \frac{2 \cdot a \cdot (a-1)}{2-a} - 2 \cdot a \cdot (k - k_0) \end{bmatrix} \quad (19)$$

As $k \to +\infty$, $k - k_0 \to +\infty$, so $\lim_{k\to+\infty} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \notin I^2$. Hence, there does not exist $k_0 \in Z^+$ such that $s_k = a$ for $k \geq k_0$, and this proves the theorem. ■

Example 4

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.7826 \\ 0.5242 \end{bmatrix}$. Figure 5 shows the phase portrait of the system.

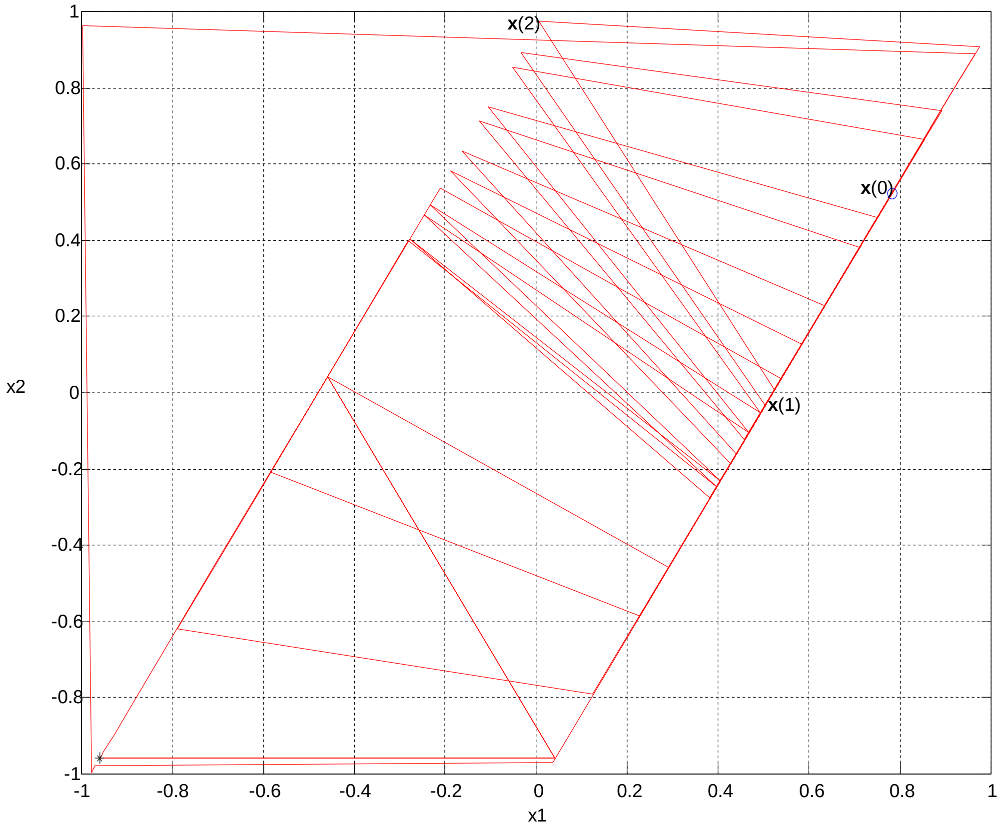

Figure 5. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the point with '*' denotes the 'steady state' of **x**.

Observation 2

When $b = -a + 1$ and a is an odd integer, the state vector converges to a fixed point on a particular straight line $x_1 = x_2$ of the phase portrait for all the initial conditions in I^2.

To demonstrate this phenomenon, a random initial condition $\mathbf{x}(0)$ is generated in I^2, it can be shown in figure 6a that the state converges to a fixed point.

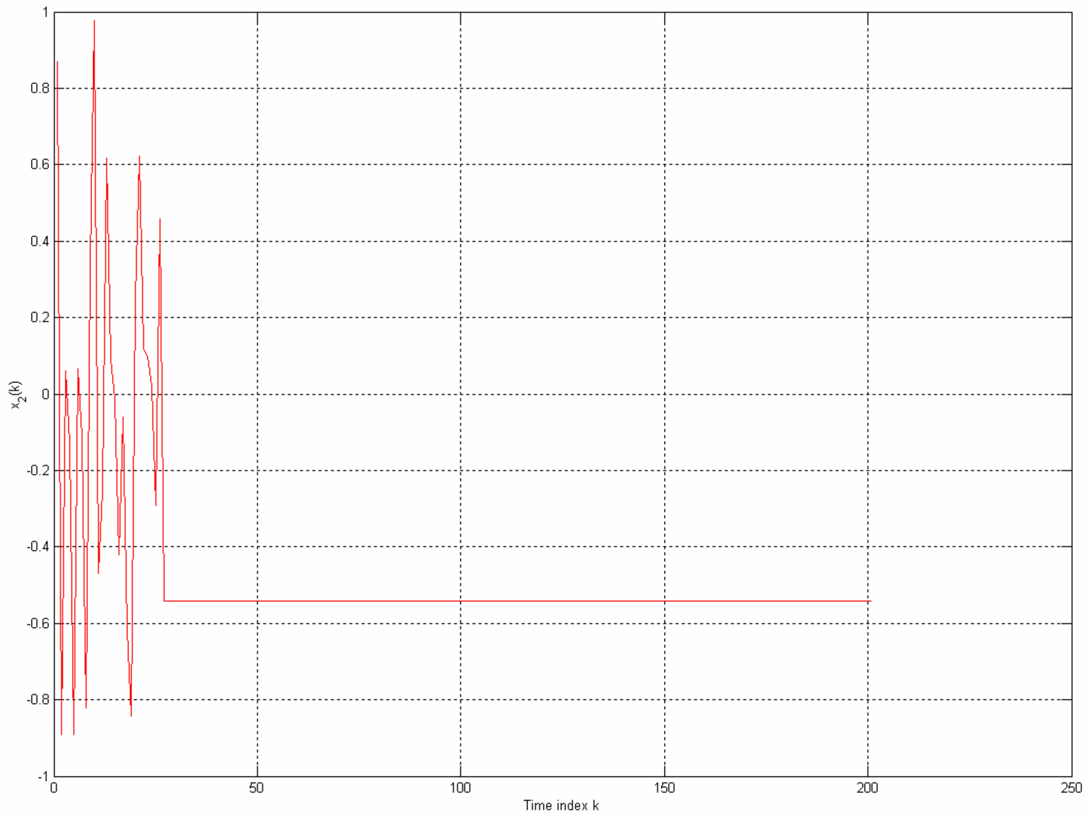

Figure 6a. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[-0.1886\ 0.87909]^T$, a=5 and b=-4, the state converges to a fixed point.

However, when a deviates from an odd integer a little bit, the state does not converge to a fixed point, as shown in figure 6b.

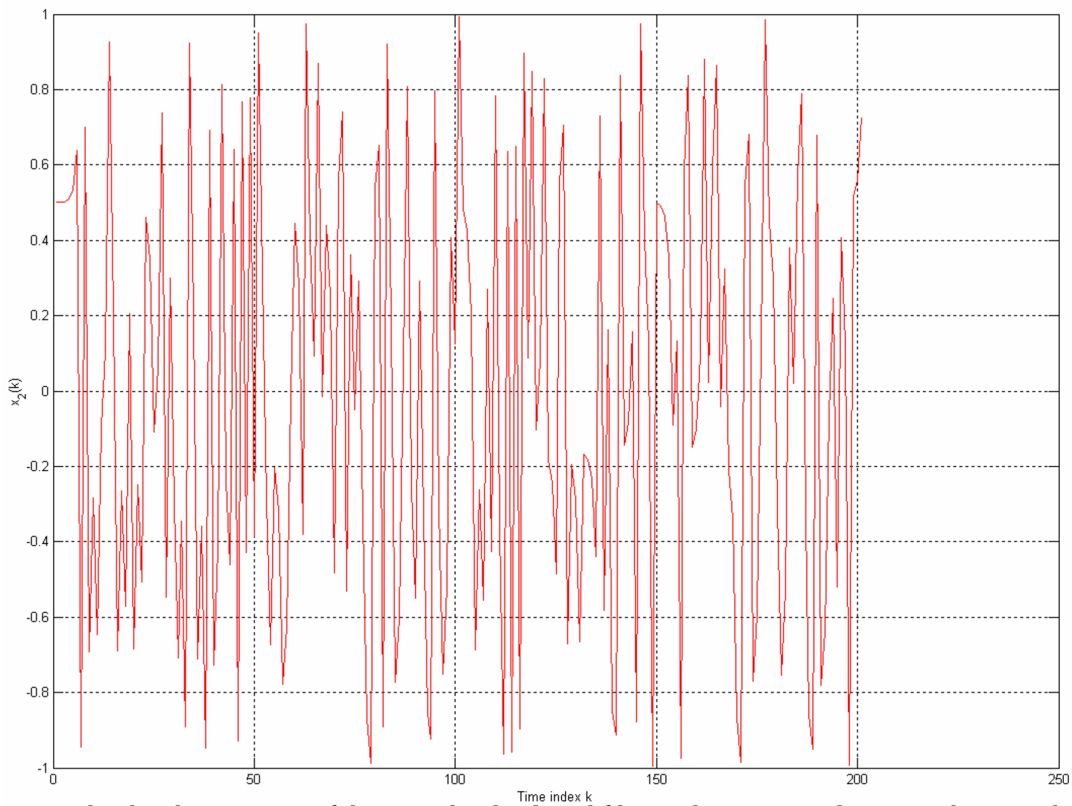

Figure 6b. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[0.5\ 0.5001]^T$, a=5.01 and b=-4.01, the state does not converge to a fixed point.

4. CONCLUSIONS

In this article, some interesting behaviors of second-order digital filters with two's complement arithmetic are explored. The cases of $b = a + 1$ and $b = -a + 1$ are analyzed and some necessary conditions for the symbolic sequences to be periodic after a number of iterations are given. The necessary and sufficient conditions for the system to behave as a linear system after a number of iterations and the state vector to toggle among several states or converge to a fixed point are given.

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