Oscillation and convergence behaviors exhibited in an 'unstable' second-order digital filter with saturation-type nonlinearity

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SUMMARY

This letter explains the oscillatory behaviors exhibited in a second-order digital filter with saturation-type nonlinearity via the Hopf bifurcation theorem. It is shown that depending on the bifurcation parameter, the state variables may converge to zero even when the eigenvalues of the system matrix are outside the unit circle.

KEY WORDS: second-order digital filter; saturation-type nonlienarity; Hopf bifurcation theorem

1. INTRODUCTION

It was reported in [1] that a second-order digital filter with saturation-type nonlinearity exhibits oscillation behaviors with different frequencies at different filter parameters when the eigenvalues of the system matrix are outside the unit circle. However, no analytical explanation is provided in [1] to account for the simulation results. The Hopf bifurcation theorem, which has been found to be very useful to account for the oscillation behaviors exhibited in a nonlinear system [2]-[5], is applied in this letter to explain the reported results of [1].

Since the eigenvalues of the system matrix are outside the unit circle, one may expect that the state trajectories will not converge to the origin. However, is this intuitive implication true? If this is not true, then under what conditions will the trajectory converge to the origin? This letter will also address this issue.

The outline of this letter is as follows: The second-order digital filter with saturation-type nonlinearity described in [1] and the Hopf bifurcation theorem discussed in [2]-[5] are briefly summarized in section 2 and section 3, respectively. In section 4, analytical results for the oscillation behaviors exhibited in a second-order digital filter with a saturation-type nonlinearity is provided. In section 5, some interesting behaviors of the state trajectories, such as the convergence behaviors, are reported. Finally, a conclusion is summarized in section 6.

2. SYSTEM DESCRIPTION

The second-order digital filter with saturation-type nonlinearity described in [1] is briefly summarized as follows:

Let the state vector of a second-order digital filter be $\mathbf{x}(k)$ and the state equation be:

$$\mathbf{x}(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathrm{F}(\mathbf{x}(k)) \tag{1}$$

If the second-order digital filter is realized by a direct form representation, then:

$$\mathbf{x}(k+1) = \begin{bmatrix} x_2(k) \\ f_s(b \cdot x_1(k) + a \cdot x_2(k)) \end{bmatrix} \tag{2}$$

where a and b are the filter parameters, and $f_s(\cdot)$ is the saturation function defined as:

$$f_s(y) = \begin{cases} \mu \cdot y & \text{if } |y| \le \frac{1}{\mu} \\ \mathrm{sgn}(y) & \text{otherwise} \end{cases} \tag{3}$$

in which μ is the gain of the saturation function in the linear region and $\mathrm{sgn}(\cdot)$ denotes the sign function.

It has been shown in [1] that the digital filter exhibits oscillation behaviors when $\mu = 1$, $a = 0.5$ and $-1.5 \le b \le -1.2$. The oscillation frequency depends on the filter parameter b, but it is independent of the initial conditions except when $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3. REVIEW OF HOPF BIFURCATION THEOREM

The Hopf bifurcation theorem discussed in [2]-[5] is briefly summarized as follows:

Suppose a discrete-time system can be represented as follows:

$$\mathbf{x}(k+1) = \mathbf{A} \cdot \mathbf{x}(k) + \mathbf{B} \cdot \mathbf{g}(\mathbf{C} \cdot \mathbf{x}(k); \mu) \tag{4}$$

$$\mathbf{y}(k) = \mathbf{C} \cdot \mathbf{x}(k) \tag{5}$$

where $\mathbf{x}(k) \in \Re^n$, $\mathbf{A} \in \Re^{nxn}$, $\mathbf{B} \in \Re^{nxl}$, $\mathbf{C} \in \Re^{mxn}$, $\mathbf{y}(k) \in \Re^m$, $\mathbf{g}(\cdot): \Re^m \times \Re \to \Re^l$ is a smooth $(C^r, r \geq 3)$ l-dimensional vector field, $k \in Z^+ \cup \{0\}$ is the iteration index, and $\mu \in \Re$ is the bifurcation parameter. It is worth to note that all the matrices may depend on μ, and $\mathbf{A}$ may be a zero matrix. By introducing an arbitrary matrix $\mathbf{D} \in \Re^{lxm}$, which may also depend on μ, the system represented by the equations (4) and (5) can also be represented by a feedback system. The whole feedback system consists of the linear feedforward part $\mathbf{G}(z;\mu)$ and the nonlinear feedback part $\mathbf{f}(\mathbf{e}(k);\mu)$ as shown in figure 1 and the system is described as below:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{y}(k)+\mathbf{B}\cdot(\mathbf{g}(\mathbf{C}\cdot\mathbf{x}(k);\mu)-\mathbf{D}\cdot\mathbf{y}(k)) \\ &= (\mathbf{A}+\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{C})\cdot\mathbf{x}(k)+\mathbf{B}\cdot(\mathbf{g}(\mathbf{C}\cdot\mathbf{x}(k);\mu)-\mathbf{D}\cdot\mathbf{y}(k))\end{aligned} \tag{6}$$

$$\mathbf{G}(z;\mu)=\mathbf{C}\cdot(z\cdot\mathbf{I}-(\mathbf{A}+\mathbf{B}\cdot\mathbf{D}\cdot\mathbf{C}))^{-1}\cdot\mathbf{B} \tag{7}$$

$$\mathbf{u}(k)=\mathbf{f}(\mathbf{e}(k);\mu)+\mathbf{v}(k)=\mathbf{g}(\mathbf{y}(k);\mu)-\mathbf{D}\cdot\mathbf{y}(k) \tag{8}$$

$$\mathbf{e}(k)=\mathbf{d}(k)-\mathbf{y}(k) \tag{9}$$

As shown in figure 1, for the autonomous system under our investigation, the reference input $\mathbf{v}(k)=\mathbf{0}$. Also, for simplicity, the external disturbance $\mathbf{d}(k)$ is assumed zero. Suppose the system achieves equilibrium at $\mathbf{e}(k)=\hat{\mathbf{e}}$. By linearizing the nonlinear feedback system $\mathbf{f}(\mathbf{e}(k);\mu)$ at the equilibrium point, the open-loop gain of the system is $\mathbf{G}(z;\mu)\cdot\mathbf{J}(\mu)$, where $\mathbf{J}(\mu)\equiv\left.\frac{\partial\mathbf{f}(\mathbf{e}(k))}{\partial\mathbf{e}(k)}\right|_{\mathbf{e}(k)=\hat{\mathbf{e}}}$ is the Jacobian matrix.

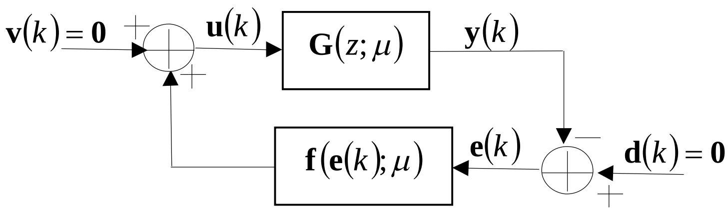

Figure 1: Nonlinear feedback system

4. ANALYTICAL RESULTS FOR THE OSCILLATION BEHAVIORS EXHIBITED IN A SECOND-ORDER DIGITAL FILTER WITH SATURATION-TYPE NONLINEARITY

For a second-order digital filter with saturation-type nonlinearity, we have:

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ b & a\end{bmatrix} \tag{10}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{11}$$

$$\mathbf{C} = \begin{bmatrix} b & a \end{bmatrix} \tag{12}$$

$$D = 0 \tag{13}$$

$$G(z) = \frac{a \cdot z + b}{z^2 - a \cdot z - b} \tag{14}$$

$$g(y(k); \mu) = \begin{cases} (\mu - 1) \cdot y(k) & |y(k)| \le \frac{1}{\mu} \\ -y(k) + \operatorname{sgn}(y(k)) & \text{otherwise} \end{cases} \tag{15}$$

and

$$f(e(k); \mu) = \begin{cases} (1 - \mu) \cdot e(k) & |e(k)| \le \frac{1}{\mu} \\ e(k) - \operatorname{sgn}(e(k)) & \text{otherwise} \end{cases} \tag{16}$$

However, $f(e(k); \mu)$ is not a smooth $(C^r, r \ge 3)$ function because $f(e(k); \mu)$ is not differentiable at $e(k) = \pm \frac{1}{\mu}$. Hence, the Hopf bifurcation theorem cannot be applied directly. However, according to [4], by letting

$$\tilde{f}(e(k); \mu) = e(k) - \frac{2}{\pi} \cdot \tan^{-1}\left(\frac{\mu \cdot \pi \cdot e(k)}{2} \right) \tag{17}$$

it can be shown easily that $\tilde{f}(e(k); \mu)$ is a smooth $(C^r, r \ge 3)$ function and $\exists \varepsilon(\mu)$ such that $\left| \tilde{f}(e(k); \mu) - f(e(k); \mu) \right| < \varepsilon(\mu)$ for $\forall e(k)$.

By solving the equation

$$G(z) \cdot \tilde{f}(e(k); \mu)\Big|_{z=1, e(k)=\hat{e}} = -e(k)\Big|_{e(k)=\hat{e}} \tag{18}$$

the equilibrium point is located at $\hat{e} = 0$. Since

$$J(\mu) = \left. \frac{\partial \tilde{f}(e(k); \mu)}{\partial e(k)} \right|_{e(k)=\hat{e}} = 1 - \mu \tag{19}$$

$$\Rightarrow \lambda(e^{j\cdot\omega}; \mu) = G(z) \cdot J(\mu)\Big|_{z=e^{j\cdot\omega}} = \frac{a \cdot e^{j\cdot\omega} + b}{e^{2\cdot j\cdot\omega} - a \cdot e^{j\cdot\omega} - b} \cdot (1 - \mu) \tag{20}$$

where $\lambda(e^{j\cdot\omega}; \mu)$ is the eigenvalue of $G(e^{j\cdot\omega}) \cdot J(\mu)$. As $G(e^{j\cdot\omega}) \cdot J(\mu)$ is a scalar, the right and left eigenvectors are:

$$\eta_1 = 1 \tag{21}$$

and

$$\eta_2 = 1 \tag{22}$$

respectively. Define:

$$H(z) \equiv \frac{G(z)}{1 + G(z) \cdot J(\mu_0)} = \frac{a \cdot z + b}{z^2 - a \cdot \mu_0 \cdot z - \mu_0 \cdot b} \tag{23}$$

$$Q \equiv \left. \frac{\partial^2 \tilde{f}(e(k); \mu)}{\partial e(k)^2} \right|_{e(k)=\hat{e}, \mu=\mu_0} \cdot \eta_1 = 0 \tag{24}$$

$$v^0 \equiv -\frac{1}{4} \cdot H\left(e^{j \cdot 0}\right) \cdot Q \cdot \overline{\eta_1} = 0 \tag{25}$$

$$v^2 \equiv -\frac{1}{4} \cdot H\left(e^{j \cdot 2 \cdot \omega_R}\right) \cdot Q \cdot \eta_1 = 0 \tag{26}$$

$$L \equiv \left. \frac{\partial^3 \tilde{f}(e(k); \mu)}{\partial e(k)^3} \right|_{e(k)=\hat{e}, \mu=\mu_0} \cdot \eta_1 \cdot \eta_1 = \frac{\pi^2 \cdot \mu_0{}^3}{2} \tag{27}$$

$$p\left(e^{j \cdot \omega_R}\right) \equiv Q \cdot v^0 + \frac{1}{2} \cdot \overline{Q} \cdot v^2 + \frac{1}{8} \cdot L \cdot \overline{\eta_1} = \frac{\pi^2 \cdot \mu_0{}^3}{16} \tag{28}$$

$$\zeta\left(e^{j \cdot \omega_R}\right) \equiv \frac{-\eta_1 \cdot G\left(e^{j \cdot \omega_R}\right) \cdot p\left(e^{j \cdot \omega_R}\right)}{\eta_1 \cdot \eta_2} = -\left(\frac{a \cdot e^{j \cdot \omega_R} + b}{e^{2 \cdot j \cdot \omega_R} - a \cdot e^{j \cdot \omega_R} - b}\right) \cdot \left(\frac{\pi^2 \cdot \mu_0{}^3}{16}\right) \tag{29}$$

where $\overline{\eta_2}$ and $\overline{Q}$ are the complex conjugates of η_2 and Q, respectively, ω_R and μ_0 are the frequency and the bifurcation parameter such that $\text{Im}\left(\lambda\left(e^{j \cdot \omega_R}; \mu_0\right)\right) = 0$. It can be easily shown that

$$\omega_R = \cos^{-1}\left(-\frac{a}{2 \cdot b}\right) \tag{30}$$

and

$$\mu_0 = 1 + \frac{e^{2 \cdot j \cdot \omega_R} - a \cdot e^{j \cdot \omega_R} - b}{a \cdot e^{j \cdot \omega_R} + b} \tag{31}$$

respectively. According to [2]-[5], the frequency of oscillations of the second-order digital filter with saturation-type nonlinearity can be approximated by the frequency at the point $\hat{P}$ where $\hat{P}$ is the intersecting point of the locus of $\lambda\left(e^{j \cdot \omega}; \mu_0\right)$ and half-line starting at $-1 + 0 \cdot j$ in the direction defined by $\zeta\left(e^{j \cdot \omega_R}\right)$. In our case, since $\zeta\left(e^{j \cdot \omega_R}\right) \in \Re$, $\hat{P}$ is the point where the locus of $\lambda\left(e^{j \cdot \omega}; \mu_0\right)$ cuts the imaginary axis. Hence, the frequency of oscillations can be approximated by ω_R.

Figure 2a shows the actual oscillation frequencies of a second-order digital filter with saturation-type nonlinearity when $\mu = 1.0001 \cdot \mu_0$, $a = 0.5$, $\mathbf{x}(0) = \begin{bmatrix} 0.0001 \\ 0.0001 \end{bmatrix}$ and $-1.5 \le b \le -1.2$, while figure 2b shows the estimated frequencies via the Hopf bifurcation theorem. It can be seen from the figure that the difference between the actual frequencies and the estimated frequencies are negligible, and so the estimations are valid.

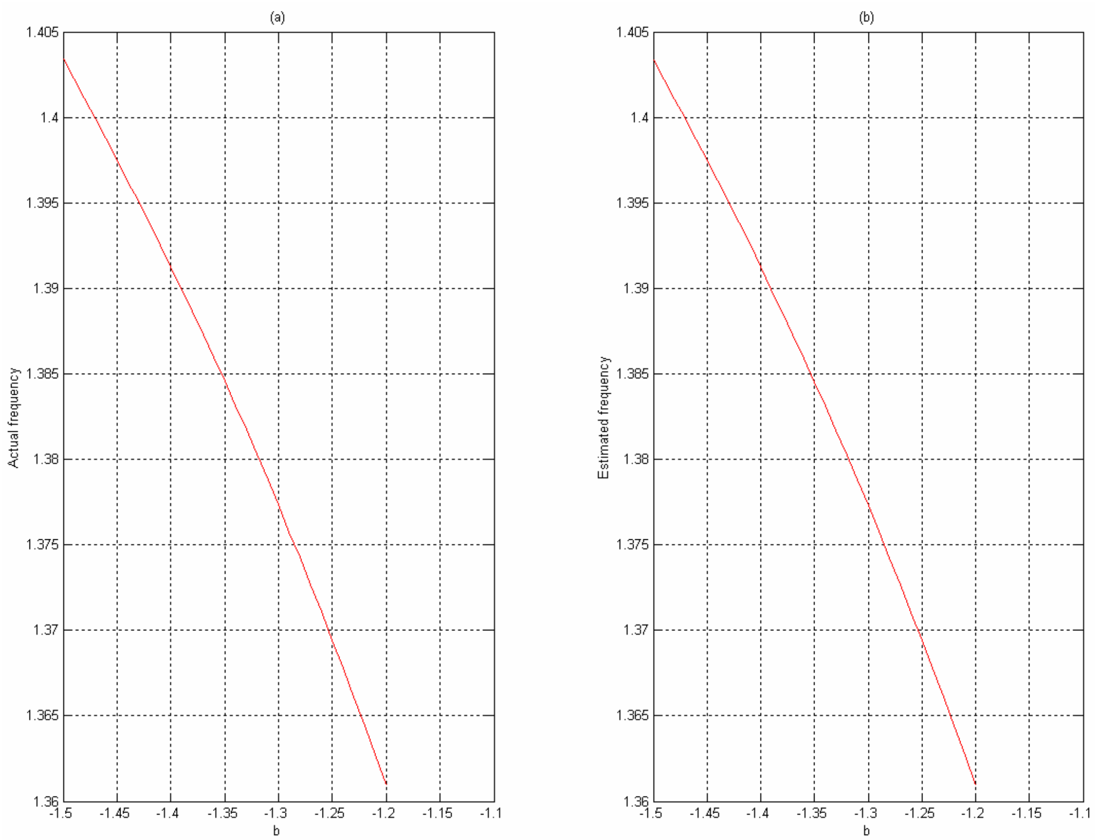

Figure 2. (a) Actual oscillation frequencies of the second-order digital filter with saturation-type nonlinearity. (b) Estimated frequencies via Hopf bifurcation theorem.

5. CONVERGENCE BEHAVIORS OF AN 'UNSTABLE' SECOND-ORDER DIGITAL FILTER WITH SATURATION-TYPE NONLINEARITY

When $\mu < \mu_0$ (before criticality), the curve $\lambda(e^{j\cdot\omega};\mu)$ encircles the point $-1+0\cdot j$ in a clockwise direction as shown in figure 3a. Hence, the overall system is stable. As a result, oscillation behaviors do not occur and the state variables converge asymptotically to the origin, even though the eignevalues of the system matrix are outside the unit circle, as shown in figure 3b.

However, when $\mu > \mu_0$ (after criticality), the curve does not encircle the point $-1+0\cdot j$ as shown in figure 3c. Hence, the equilibrium point is unstable. As a result, the state trajectories will move from the neighbor of the equilibrium point to the limit cycle and oscillation occurs, as shown in figure 3d. In this case, the limit cycle is stable.

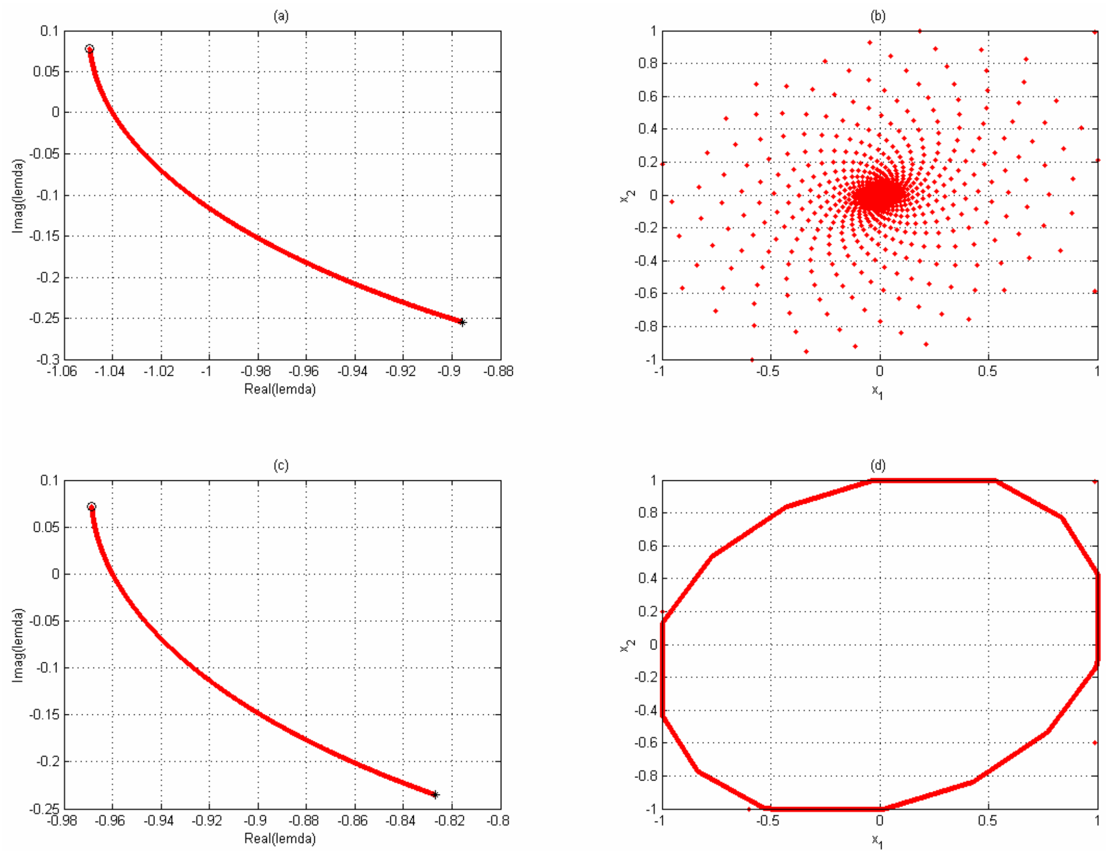

Figure 3. (a) The curve $\lambda\left(e^{j\cdot\omega};\mu\right)$ when $\mu = 0.99\cdot\mu_0$, $a = 0.5$, $b = -1.25$ and $\mathbf{x}(0) = \begin{bmatrix} 0.99 \\ 0.99 \end{bmatrix}$. (b) The corresponding state trajectories. (c) The curve $\lambda\left(e^{j\cdot\omega};\mu\right)$ when $\mu = 1.01\cdot\mu_0$, $a = 0.5$, $b = -1.25$ and $\mathbf{x}(0) = \begin{bmatrix} 0.99 \\ 0.99 \end{bmatrix}$. (d) The corresponding state trajectories.

6. CONCLUSION

In this letter, the Hopf bifurcation theorem is employed to account for the oscillation behaviors exhibited in a second-order digital filter with saturation-type nonlinearity. It is also found that the state trajectories may converge to the origin even though the eigenvalues of the system matrix are outside the unit circle.

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