Autonomous Response of a Third-order Digital Filter with Two's Complement Arithmetic Realized in Cascade Form

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Abstract

In this letter, results on the autonomous response of a third-order digital filter with two's complement arithmetic realized as a first-order subsystem cascaded by a second-order subsystem are reported. The behavior of the second-order subsystem depends on the pole location and the initial condition of the first-order subsystem, because the transient behavior is affected by the first-order subsystem and this transient response can be viewed as an excitation of the original initial state to another state. New results on the set of necessary and sufficient conditions relating the trajectory equations, the behaviors of the symbolic sequences, and the sets of the initial conditions are derived. The effects of the pole location and the initial condition of first-order subsystem on the overall system are discussed. Some interesting differences between the autonomous response of second-order subsystem and the response due to the exponentially decaying input are reported. Some simulation results are given to illustrate the analytical results.

1. Introduction

It is well known that the autonomous response of a marginally stable second-order digital filter with two's complement arithmetic may exhibit chaotic behaviors, dependent on the initial conditions [1], [4], [6], [7], [10]. Similar behaviors are reported for the stable and unstable cases [8], [9], [12], [14]. Investigations on the chaotic behaviors of filters with saturation-type nonlinearity and quantization-type nonlinearity have been discussed in [2] and [5], [11], respectively.

The step response and sinusoidal response of a marginally stable second-order digital filter with two's complement arithmetic are studied in [15], [16], respectively. It is found that the trajectory of the step response case is similar to the autonomous response case, while that of the sinusoidal response case is more complicated.

In [3], a marginally stable third-order digital filter with two's complement arithmetic implemented in direct form is analyzed. It is found that the trajectory is on some planes of the phase portrait. By realizing the third-order digital filter in parallel form and plotting the delayed output $y(k-1)$ versus the output $y(k)$, some related patterns are reported in [13].

A third-order linear digital filter can be realized as a first-order subsystem cascaded with a second-order subsystem, and the overall system behavior can be predicted easily. However, for a third-order digital filter with two's complement arithmetic, the second-order subsystem may exhibit chaotic behavior, the steady state behavior of the overall system may not be similar to that of the second-order subsystem even though the output of the first-order subsystem under the autonomous response case will decay exponentially to zero if its pole is inside the unit circle. This is because a very small non-zero exponentially decaying input may give an output very different from the autonomous response case [16]. Thus, we may ask the questions: Under what conditions will the overall system behave as the autonomous response of the second-order subsystem? How does this non-zero exponentially decaying signal affect the behavior of the overall system?

Although the autonomous response of the overall system may be similar to that of the second-order subsystem, the pole location and the initial condition of the first-order subsystem may affect the behavior. Basically, the transient behavior of the second-order subsystem can be viewed as that due to an excitation of the original initial state to another state, which can be taken as the effective "initial" state that governs the steady state behavior of the second-order subsystem. So, how do the initial condition and the pole location of the first-order subsystem affect the value of this effective "initial" state and the behavior of the overall system?

We will review the notations used in the existing literature in section 2 [1]-[16]. In section 3, the behavior of a third-order digital filter with two's complement arithmetic realized in cascade form is discussed. Finally, a conclusion is summarized in section 4.

2. Notations

The notations used in [1]-[16] are adopted as follows:

Assume that a third-order digital filter is represented by a first-order subsystem cascaded with a second-order subsystem realized in direct form with both subsystems implemented using two's complement arithmetic, as shown in figure 1. The state space model of the overall system can be represented as:

$$x_3(k+1) = f(c \cdot x_3(k) + u(k)), \tag{2.1}$$

$$y_1(k) = x_3(k), \tag{2.2}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} x_2(k) \\ f(b \cdot x_1(k) + a \cdot x_2(k) + y_1(k)) \end{bmatrix}, \tag{2.3}$$

and

$$y(k) = x_1(k), \tag{2.4}$$

where $x_1(k)$ and $x_2(k)$ are the state variables of the second-order subsystem;

$x_3(k)$ is the state variable of the first-order subsystem; $u(k)$ is the input signal of the overall system; $y_1(k)$ is the output of the first-order subsystem; $y(k)$ is the output of the overall system; a and b are the filter parameters of the second-order subsystem; c is the filter parameter of the first-order subsystem; f is the nonlinearity introduced by the two's complement arithmetic.

The nonlinear function f can be modeled as:

$$f(v) = v - 2\cdot n \text{ such that } 2\cdot n - 1 \le v < 2\cdot n + 1, \text{ where } n \in \mathsf{Z}. \tag{2.5}$$

Hence, $$\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \in I^3 \equiv \left\{ \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} : -1 \le x_1(k) < 1, -1 \le x_2(k) < 1 \text{ and } -1 \le x_3(k) < 1 \right\}. \tag{2.6}$$

In our analysis, we only consider the autonomous response case, that is:

$$u(k) = 0 \text{ for } k \ge 0. \tag{2.7}$$

Similar to [13], we assume that the first-order subsystem is stable or marginally stable, and the second-order subsystem is marginally stable, that is:

$$|c| \le 1, \tag{2.8}$$

$$b = -1, \tag{2.9}$$

and

$$|a| \le 2. \tag{2.10}$$

Since $|a| \le 2$, we define $$\theta \equiv \cos^{-1}\left(\frac{a}{2}\right), \tag{2.11}$$

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}, \tag{2.12}$$

$$\hat{\mathbf{A}} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{2.13}$$

and

$$\mathbf{T} \equiv \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix}, \tag{2.14}$$

then we have:

$$\mathbf{A} = \mathbf{T} \cdot \hat{\mathbf{A}} \cdot \mathbf{T}^{-1}. \tag{2.15}$$

Let $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (2.16)

Define $s_1(k)$ and $s_2(k)$ such that:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} \equiv \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot y_1(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s_1(k) \tag{2.17}$$

and

$$x_3(k+1) \equiv c \cdot x_3(k) + u(k) + 2 \cdot s_2(k). \tag{2.18}$$

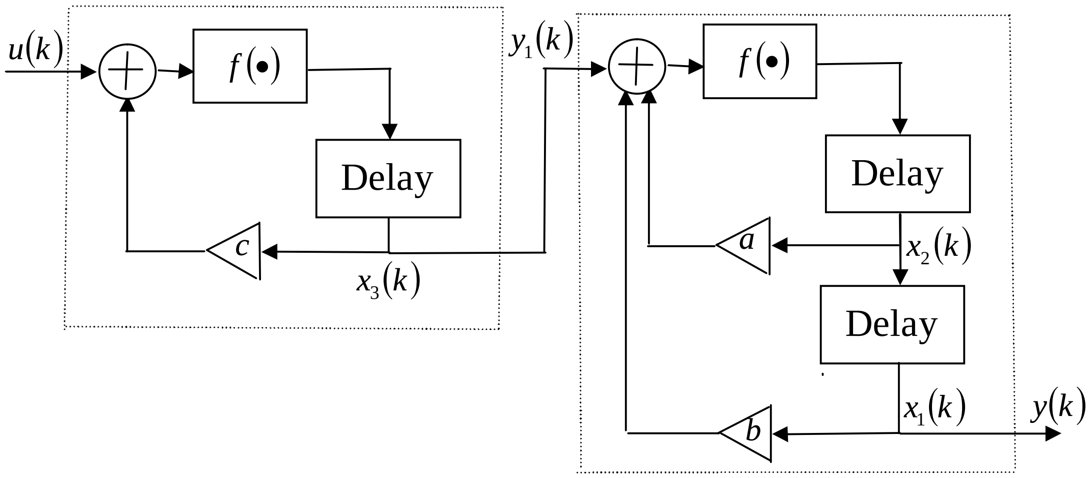

Figure 1: Cascade realization of a third-order digital filter with two's complement arithmetic

3. Analytical and simulation results

In this section, we will analyze the trajectory equations, behaviors of symbolic sequences, and the sets of initial conditions of the system described in section 2 for various types of trajectories, by considering the following three cases: $c = 1$, $c = -1$, and $|c| < 1$, respectively.

When $c = 1$, we have $x_3(k) = y_1(k) = x_3(0)$ and $s_2(k) = 0$ for $k \geq 0$. The behavior of the overall system is thus equivalent to that of the step response of the second-order subsystem. Hence, our results in [15] can be applied. When $c = -1$ and

$x_3(0)=-1$, then $y_1(k)=x_3(k)=-1$ and $s_2(k)=-1$ for $k\geq 0$. The behavior of the overall system is also equivalent to that of the step response of the second-order subsystem having the input step size equal to -1. When $c=-1$ and $x_3(0)\neq -1$, then $y_1(k)=x_3(k)=x_3(0)$ for k is even and $y_1(k)=x_3(k)=-x_3(0)$ for k is odd, while $s_2(k)=0$ for $k\geq 0$. The behavior of the overall system is equivalent to that of the sinusoidal response of the second-order subsystem with the period of the input sinusoidal signal being 2. Hence, our results in [16] can be applied.

When $|c|<1$, then $s_2(k)=0$ and $y_1(k)=x_3(k)=c^k x_3(0)$ for $k\geq 0$, and we have:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A}\cdot\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B}\cdot c^k x_3(0) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cdot s_1(k), \text{ for } k\geq 0. \tag{3.1}$$

Lemma 1 (L1):

$\exists k_0\geq 0$ such that $s_1(k)=0$ for $k\geq k_0$ *if and only if*:

$$\left\| \mathbf{T}^{-1}\cdot\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \mathbf{T}^{-1}\cdot\frac{x_3(k_0)}{c^2-2\cdot\cos\theta\cdot c+1}\cdot\begin{bmatrix} 1 \\ c \end{bmatrix} \right\| + \left| \frac{x_3(k_0)}{c^2-2\cdot\cos\theta\cdot c+1} \right| < 1. \tag{3.2}$$

Proof:

For the *if* part, if $\exists k_0\geq 0$ such that $s_1(k)=0$ for $k\geq k_0$, then equation (3.1) becomes:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \mathbf{A}^{k-k_0}\cdot\left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \frac{x_3(k_0)}{c^2-2\cdot\cos\theta\cdot c+1}\cdot\begin{bmatrix} 1 \\ c \end{bmatrix}\right) + \frac{x_3(k_0)\cdot c^{k-k_0}}{c^2-2\cdot\cos\theta\cdot c+1}\cdot\begin{bmatrix} 1 \\ c \end{bmatrix}, \tag{3.3}$$

for $k\geq k_0$.

Define $$\bar{\mathbf{x}} \equiv \frac{x_3(k_0)}{c^2-2\cdot\cos\theta\cdot c+1}\cdot\begin{bmatrix} 1 \\ c \end{bmatrix}. \tag{3.4}$$

Equation (3.3) becomes:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \mathbf{A}^{k-k_0}\cdot\left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \bar{\mathbf{x}}\right) + c^{k-k_0}\cdot\bar{\mathbf{x}}, \text{ for } k\geq k_0. \tag{3.5}$$

Define $\hat{\mathbf{x}}(k) \equiv \mathbf{T}^{-1} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$ for $k \geq 0$, (3.6)

and $\hat{\mathbf{x}}(k_0) - \mathbf{T}^{-1} \cdot \bar{\mathbf{x}} \equiv \rho \cdot \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$. (3.7)

Equation (3.5) becomes:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \rho \cdot \begin{pmatrix} \cos(\phi - (k - k_0) \cdot \theta) \\ \cos(\phi - (k - k_0 + 1) \cdot \theta) \end{pmatrix} + c^{k-k_0} \cdot \bar{\mathbf{x}}, \text{ for } k \geq k_0. \tag{3.8}$$

$|x_i(k)| < 1$ for $k \geq 0$ and $i = 1,2$ $\Rightarrow \rho + \left| \dfrac{x_3(k_0)}{c^2 - 2 \cdot \cos\theta \cdot c + 1} \right| < 1$, and this proves the *if* part.

For the *only if* part, if $\exists k_0 \geq 0$ such that:

$$\left\| \mathbf{T}^{-1} \cdot \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \mathbf{T}^{-1} \cdot \frac{x_3(k_0)}{c^2 - 2 \cdot \cos\theta \cdot c + 1} \cdot \begin{bmatrix} 1 \\ c \end{bmatrix} \right\| + \left| \frac{x_3(k_0)}{c^2 - 2 \cdot \cos\theta \cdot c + 1} \right| < 1, \tag{3.9}$$

then the trajectory equation of the corresponding linear system is:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \mathbf{A}^{k-k_0} \cdot \left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \bar{\mathbf{x}} \right) + c^{k-k_0} \cdot \bar{\mathbf{x}}, \text{ for } k \geq k_0, \tag{3.10}$$

and the state will be bounded, that is, $|x_i(k)| < 1$ for $k \geq 0$ and $i = 1,2$. (3.11)

Hence, the overall system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot c^k x_3(0) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s_1(k) \tag{3.12}$$

does not have an overflow for $k \geq k_0$, which implies that $s_1(k) = 0$ for $k \geq k_0$. This proves the *only if* part, and completes the proof. ■

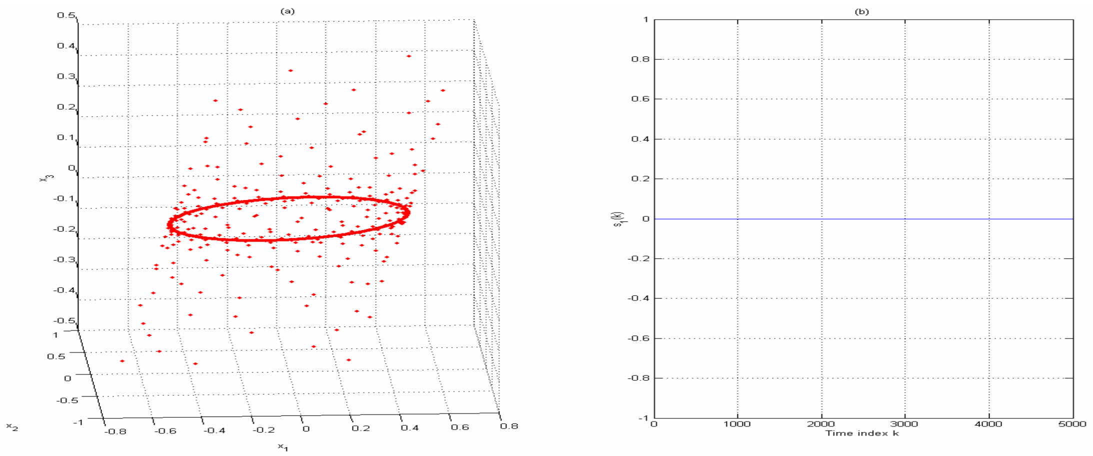

Figure 2: Behavior of type I trajectory when $x_1(0) = -0.5$, $x_2(0) = 0.5$, $x_3(0) = -0.5$, $c = -0.98$, and $a = 0.5$. (a) Trajectory of the third-order digital filter. (b) Symbolic sequence $\{s_1(k)\}$.

Lemma 2 (L2):

By defining:

$$\tilde{\mathbf{x}}_0 \equiv \left(c^M \cdot \mathbf{I} - \mathbf{A}^M\right)^{-1} \cdot \sum_{j=0}^{M-1} \mathbf{A}^{M-1-j} \cdot \mathbf{B} \cdot c^j \cdot x_3(k_0), \tag{3.13}$$

$$\tilde{\mathbf{y}}_0 \equiv \left(\mathbf{I} - \mathbf{A}^M\right)^{-1} \cdot \sum_{j=0}^{M-1} \mathbf{A}^{M-1-j} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s_1(k_0 + j), \tag{3.14}$$

$$\tilde{\mathbf{x}}_{i+1} \equiv \mathbf{A} \cdot \tilde{\mathbf{x}}_i + \mathbf{B} \cdot c^i \cdot x_3(k_0) \quad \text{for} \quad i = 0,1,\cdots,M-2, \tag{3.15}$$

and

$$\tilde{\mathbf{y}}_{i+1} \equiv \mathbf{A} \cdot \tilde{\mathbf{y}}_i + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s_1(k_0 + i) \quad \text{for} \quad i = 0,1,\cdots,M-2, \tag{3.16}$$

we have:

$\exists k_0 \geq 0$ and $\exists M \in Z^+$ such that $s_1(k) = s_1(k+M)$ for $k \geq k_0$ *if and only if*

$$\left\| \mathbf{T}^{-1} \cdot \left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \tilde{\mathbf{x}}_0 - \tilde{\mathbf{y}}_0 \right) \right\| < 1 - \left\| \tilde{\mathbf{x}}_i + \tilde{\mathbf{y}}_i \right\|_\infty, \text{ for } i = 0,1,\cdots,M-1. \tag{3.17}$$

Proof:

For the *if* part, if $\exists k_0 \geq 0$ and $\exists M \in Z^+$ such that $s_1(k) = s_1(k+M)$ for $k \geq k_0$, then:

$$\begin{bmatrix} x_1(k_0+p\cdot M) \\ x_2(k_0+p\cdot M) \end{bmatrix} = \mathbf{A}^{p\cdot M}\cdot\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} + \left(c^{p\cdot M}\cdot\mathbf{I}-\mathbf{A}^{p\cdot M}\right)\cdot\tilde{\mathbf{x}}_0 + \left(\mathbf{I}-\mathbf{A}^{p\cdot M}\right)\cdot\tilde{\mathbf{y}}_0, \text{ for } \quad p\geq 0, \tag{3.18}$$

and

$$\begin{aligned} \begin{bmatrix} x_1(k_0+p\cdot M+i) \\ x_2(k_0+p\cdot M+i) \end{bmatrix} &= \mathbf{A}^{p\cdot M+i}\cdot\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} + \left(c^{p\cdot M}\cdot\mathbf{I}-\mathbf{A}^{p\cdot M}\right)\cdot\mathbf{A}^i\cdot\tilde{\mathbf{x}}_0 + \left(\mathbf{I}-\mathbf{A}^{p\cdot M}\right)\cdot\mathbf{A}^i\cdot\tilde{\mathbf{y}}_0 \\ &+ c^{p\cdot M}\cdot x_3(k_0)\cdot\sum_{j=0}^{i-1}\mathbf{A}^{i-1-j}\cdot\mathbf{B}\cdot c^j + \sum_{j=0}^{i-1}\mathbf{A}^{i-1-j}\cdot\begin{bmatrix} 0 \\ 2 \end{bmatrix}\cdot s_1(k_0+j) \end{aligned}, \tag{3.19}$$

for $p\geq 0$ and $i=1,2,\cdots,M-1$.

This is equivalent to:

$$\begin{bmatrix} x_1(k_0+p\cdot M+i) \\ x_2(k_0+p\cdot M+i) \end{bmatrix} = \mathbf{A}^{p\cdot M+i}\cdot\left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \tilde{\mathbf{x}}_0 - \tilde{\mathbf{y}}_0\right) + c^{p\cdot M}\cdot\tilde{\mathbf{x}}_i + \tilde{\mathbf{y}}_i, \tag{3.20}$$

for $p\geq 0$ and $i=0,1,\cdots,M-1$.

By defining:

$$\mathbf{T}^{-1}\cdot\left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \tilde{\mathbf{x}}_0 - \tilde{\mathbf{y}}_0\right) \equiv \rho'\cdot\begin{bmatrix} \cos\phi' \\ \sin\phi' \end{bmatrix}, \tag{3.21}$$

then:

$$\begin{bmatrix} x_1(k_0+p\cdot M+i) \\ x_2(k_0+p\cdot M+i) \end{bmatrix} = \rho'\cdot\begin{bmatrix} \cos(\phi'-(p\cdot M+i)\cdot\theta) \\ \cos(\phi'-(p\cdot M+i+1)\cdot\theta) \end{bmatrix} + c^{p\cdot M}\cdot\tilde{\mathbf{x}}_i + \tilde{\mathbf{y}}_i, \tag{3.22}$$

for $p\geq 0$ and $i=0,1,\cdots,M-1$.

Since, $|x_j(k)|<1$ for $k\geq 0$ and $j=1,2$, we have:

$$\left\|\mathbf{T}^{-1}\cdot\left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \tilde{\mathbf{x}}_0 - \tilde{\mathbf{y}}_0\right)\right\| < 1 - \left\|\tilde{\mathbf{x}}_i + \tilde{\mathbf{y}}_i\right\|_\infty, \tag{3.23}$$

for $i=0,1,\cdots,M-1$, and this proves the *if* part.

For the *only if* part, since:

$$\tilde{\mathbf{y}}_i = \mathbf{A}^i\cdot\tilde{\mathbf{y}}_0 + \sum_{j=0}^{i-1}\mathbf{A}^{i-1-j}\cdot\begin{bmatrix} 0 \\ 2 \end{bmatrix}\cdot s_1(k_0+j), \text{ for } i=1,\cdots,M-1, \tag{3.24}$$

and $\left(\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{y}}_0=\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot s_1\left(k_0+j\right),$ (3.25)

we have:

$$\left(\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{y}}_i=\sum_{j=0}^{M-1-i}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot s_1\left(k_0+i+j\right)+\sum_{j=M-i}^{M-1}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot s_1\left(k_0+i+j-M\right),\quad(3.26)$$

for $i=1,\cdots,M-1$.

Similarly, since $\tilde{\mathbf{x}}_i=\mathbf{A}^{i}\cdot\tilde{\mathbf{x}}_0+\sum_{j=0}^{i-1}\mathbf{A}^{i-1-j}\cdot\mathbf{B}\cdot c^{j}\cdot x_3\left(k_0\right)$, for $i=1,\cdots,M-1$, (3.27)

and $\left(c^{M}\cdot\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{x}}_0=\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\cdot\mathbf{B}\cdot c^{j}\cdot x_3\left(k_0\right),$ (3.28)

we have:

$\left(c^{M}\cdot\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{x}}_i=\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\cdot\mathbf{B}\cdot c^{i+j}\cdot x_3\left(k_0\right)$, for $i=0,\cdots,M-1$. (3.29)

As:

$$\begin{bmatrix}x_1\left(k_0+(p+1)\cdot M+i\right)\\x_2\left(k_0+(p+1)\cdot M+i\right)\end{bmatrix}=\mathbf{A}^{M}\cdot\begin{bmatrix}x_1\left(k_0+p\cdot M+i\right)\\x_2\left(k_0+p\cdot M+i\right)\end{bmatrix}+\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\cdot\mathbf{B}\cdot c^{p\cdot M+i+j}\cdot x_3\left(k_0\right)$$
$$+\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot s_1\left(k_0+p\cdot M+i+j\right),\quad(3.30)$$

for $i=0,1,\cdots,M-1$ and $p\geq 0$,

by letting $\hat{\mathbf{x}}_i(p)=\mathbf{T}^{-1}\cdot\begin{bmatrix}x_1\left(k_0+p\cdot M+i\right)\\x_2\left(k_0+p\cdot M+i\right)\end{bmatrix}$, for $i=0,1,\cdots,M-1$ and $p\geq 0$, (3.31)

we have:

$$\hat{\mathbf{x}}_0(p+1)=\hat{\mathbf{A}}^{M}\cdot\hat{\mathbf{x}}_0(p)+c^{p\cdot M}\cdot\mathbf{T}^{-1}\cdot\left(c^{M}\cdot\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{x}}_0$$
$$+\mathbf{T}^{-1}\cdot\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot\left(s_1\left(k_0+p\cdot M+j\right)-s_1\left(k_0+j\right)\right)+\mathbf{T}^{-1}\cdot\left(\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{y}}_0,\quad(3.32)$$

for $p\geq 0$, and

$$\hat{\mathbf{x}}_i(p+1)=\hat{\mathbf{A}}^{M}\cdot\hat{\mathbf{x}}_i(p)+c^{p\cdot M}\cdot\mathbf{T}^{-1}\cdot\left(c^{M}\cdot\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{x}}_i$$
$$+\mathbf{T}^{-1}\cdot\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot s_1\left(k_0+p\cdot M+i+j\right)+\mathbf{T}^{-1}\cdot\left(\mathbf{I}-\mathbf{A}^{M}\right)\cdot\tilde{\mathbf{y}}_i\quad,\quad(3.33)$$
$$-\mathbf{T}^{-1}\cdot\sum_{j=0}^{M-1-i}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot s_1\left(k_0+i+j\right)-\mathbf{T}^{-1}\cdot\sum_{j=M-i}^{M-1}\mathbf{A}^{M-1-j}\cdot\begin{bmatrix}0\\2\end{bmatrix}\cdot s_1\left(k_0+i+j-M\right)$$

for $i = 1, \cdots, M-1$ and $p \ge 0$.

When $p = 0$, we have:

$$\hat{\mathbf{x}}_i(1) = \hat{\mathbf{A}}^M \cdot \hat{\mathbf{x}}_i(0) + \mathbf{T}^{-1} \cdot \left(c^M \cdot \mathbf{I} - \mathbf{A}^M\right) \cdot \tilde{\mathbf{x}}_i + \mathbf{T}^{-1} \cdot \left(\mathbf{I} - \mathbf{A}^M\right) \cdot \tilde{\mathbf{y}}_i + \mathbf{T}^{-1} \cdot \sum_{j=0}^{i-1} \mathbf{A}^{i-1-j} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot \left(s_1(k_0 + M + j) - s_1(k_0 + j)\right), \text{ for } i = 1, \cdots, M-1. \quad (3.34)$$

If $\left\| \mathbf{T}^{-1} \cdot \left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \tilde{\mathbf{x}}_0 - \tilde{\mathbf{y}}_0 \right) \right\| < 1 - \left\| \tilde{\mathbf{x}}_i + \tilde{\mathbf{y}}_i \right\|_\infty$ for $i = 0, 1, \cdots, M-1$, then:

for $i = 1$, we have $s_1(k_0 + M) = s_1(k_0)$. (3.35)

For $i = 2$, since $s_1(k_0 + M) = s_1(k_0)$, we have $s_1(k_0 + M + 1) = s_1(k_0 + 1)$. (3.36)

Similarly, we have $s_1(k_0 + M + j) = s_1(k_0 + j)$ for $j = 0, 1, \cdots, M-2$. (3.37)

When $p = 1$, we have:

$$\hat{\mathbf{x}}_0(2) = \hat{\mathbf{A}}^M \cdot \hat{\mathbf{x}}_0(1) + c^M \cdot \mathbf{T}^{-1} \cdot \left(c^M \cdot \mathbf{I} - \mathbf{A}^M\right) \cdot \tilde{\mathbf{x}}_0 + \mathbf{T}^{-1} \cdot \sum_{j=0}^{M-1} \mathbf{A}^{M-1-j} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot \left(s_1(k_0 + M + j) - s_1(k_0 + j)\right) + \mathbf{T}^{-1} \cdot \left(\mathbf{I} - \mathbf{A}^M\right) \cdot \tilde{\mathbf{y}}_0. \quad (3.38)$$

Since $s_1(k_0 + M + j) = s_1(k_0 + j)$ for $j = 0, 1, \cdots, M-2$, we have:

$s_1(k_0 + 2 \cdot M - 1) = s_1(k_0 + M - 1)$. (3.39)

Similarly, we have $s_1(k_0 + p \cdot M + i) = s_1(k_0 + i)$, for $i = 0, \cdots, M-1$ and $p \ge 0$. This proves the *only if* part, and completing the proof. ■

Discussion on L2: Similar to Lemma 1, Lemma 2 gives a set of necessary and sufficient conditions relating the trajectory equation, behavior of the symbolic sequence $\{s_1(k)\}$, and the set of initial condition for which the overall system exhibits the type II trajectory.

Since $x_3(k) = c^k x_3(0)$ for $k \ge 0$, the trajectory will converge to the horizontal plane $x_3 = 0$. According to equation (3.22), if k is large enough that the term $c^{p \cdot M} \cdot \tilde{\mathbf{x}}_i$, for $i = 0, 1, \cdots, M-1$, can be neglected, then

$\begin{bmatrix} x_1(k_0 + p \cdot M + i) \\ x_2(k_0 + p \cdot M + i) \end{bmatrix} \approx \rho' \cdot \begin{bmatrix} \cos(\phi' - (p \cdot M + i) \cdot \theta) \\ \cos(\phi' - (p \cdot M + i + 1) \cdot \theta) \end{bmatrix} + \tilde{\mathbf{y}}_i$, for $i = 0,1,\cdots,M-1$. This trajectory equation corresponds to M ellipses centered at $\tilde{\mathbf{y}}_i$, for $i = 0,1,\cdots,M-1$, as shown in figure 3a. These centers only depend on the filter parameter of the second-order subsystem and the corresponding M distinct symbolic sequences $\{s_1(k_0 + i)\}$, for $i = 0,1,\cdots,M-1$. By using simple transformations, these ellipses can be transformed to a circle centered at the origin, and the radius of the circle is ρ'. Hence, the sizes of these ellipses are equal and dependent on both the initial condition and the filter parameters of both the first-order and second-order subsystems. The orientation of these ellipses depend only on the filter parameter a of the second-order subsystem.

Similar to the type I trajectory, $x_i(k)$, for $i = 1,2,3$, is never periodic no matter whether θ is a rational multiple of π or not, because the term $c^k x_3(0)$ and the terms $c^{p \cdot M} \cdot \tilde{\mathbf{x}}_i$, for $i = 0,1,\cdots,M-1$, are never periodic. Hence, the spectrum of $x_i(k)$, for $i = 1,2,3$, is continuous.

From Lemma 2, we can conclude that the overall system exhibits type II trajectory if and only if the symbolic sequence $\{s_1(k)\}$ is periodic with period M for $k \geq k_0$, as shown in figure 3b. From equation (3.17), the set of initial conditions that gives the type II trajectory also looks like many inclined cyclones.

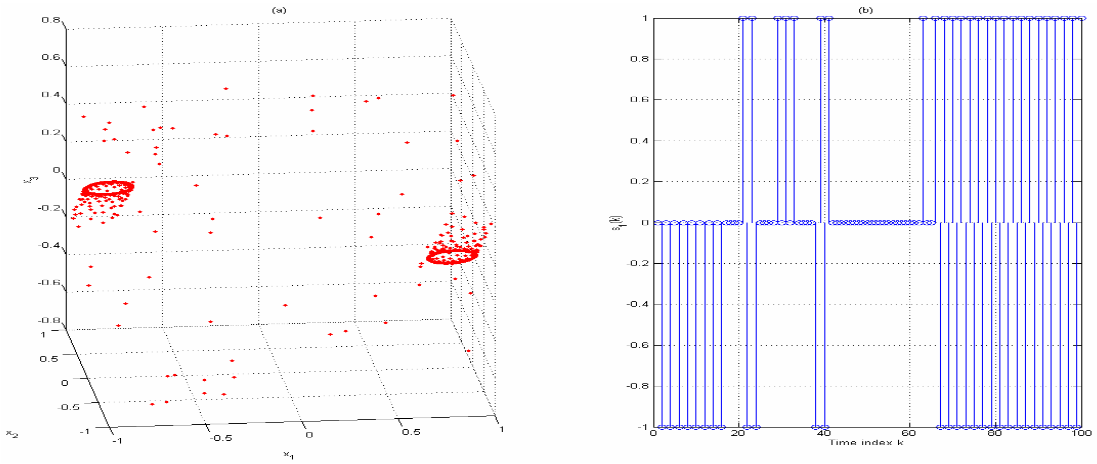

Figure 3: Behavior of type II trajectory when $x_1(0) = -0.75$, $x_2(0) = -0.75$, $x_3(0) = -0.75$, $c = -0.98$, and $a = 0.5$. (a) Trajectory of the third-order digital filter. (b) Symbolic sequence $\{s_1(k)\}$.

Lemma 3 (L3):

Define:

$$\Xi'_0 = \left\{ \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \\ x_3(k_0) \end{bmatrix} : \left\| \mathbf{T}^{-1} \cdot \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \mathbf{T}^{-1} \cdot \frac{x_3(k_0)}{c^2 - 2 \cdot \cos\theta \cdot c + 1} \cdot \begin{bmatrix} 1 \\ c \end{bmatrix} \right\| + \left| \frac{x_3(k_0)}{c^2 - 2 \cdot \cos\theta \cdot c + 1} \right| < 1 \right\} \tag{3.40}$$

$$\Xi'_1 = \left\{ \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \\ x_3(k_0) \end{bmatrix} : \left\| \mathbf{T}^{-1} \cdot \left(\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} - \tilde{\mathbf{x}}_0 - \tilde{\mathbf{y}}_0 \right) \right\| < 1 - \left\| \tilde{\mathbf{x}}_i + \tilde{\mathbf{y}}_i \right\|_\infty, \text{ for } i = 0,1,\cdots M-1 \right\} \tag{3.41}$$

$$\Xi'_2 \equiv I^3 \setminus (\Xi'_0 \cup \Xi'_1). \tag{3.42}$$

The following three statements for the type III trajectory are equivalent:

(L3.1) The trajectory of $\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$ will converge to a horizontal plane $x_3 = 0$ and exhibits an elliptical fractal pattern on the plane $x_3 = 0$, (3.43)

(L3.2) $s_1(k)$ is aperiodic for $k \geq k_0$, (3.44)

and

$$\text{(L3.3)} \quad \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \\ x_3(k_0) \end{bmatrix} \in \Xi_2' . \tag{3.45}$$

According to extensive simulations, we found that Lemma 3 is true. Figure 4a and figure 4b show an example trajectory of a third-order digital filter with two's complement arithmetic realized in cascade form and its corresponding symbolic sequence $\{s_1(k)\}$, respectively.

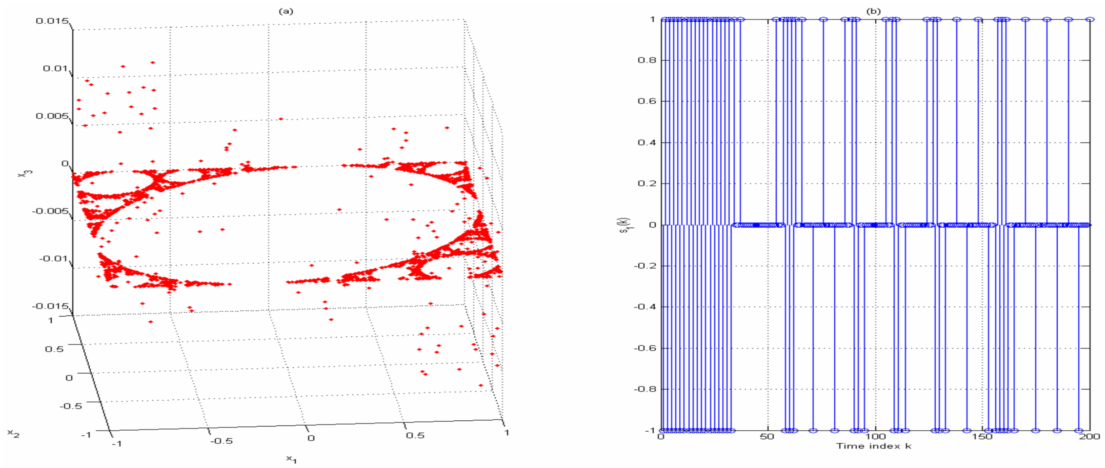

Figure 4: Behavior of type III trajectory when $x_1(0) = -0.6$, $x_2(0) = 0.9$, $x_3(0) = 0.012$, $c = -0.98$, and $a = 0.5$. (a) Trajectory of the third-order digital filter. (b) Symbolic sequence $\{s_1(k)\}$.

4. Conclusion

In this letter, the autonomous response of a third-order digital filter with two's complement arithmetic realized in cascade form is investigated. The main contribution of this letter is to derive sets of necessary and sufficient conditions relating the trajectory equations, behaviors of symbolic sequence, and the sets of initial conditions that give various types of trajectories. Based on these necessary and sufficient conditions, the exponentially decaying input fed into the second-order subsystem will give an output for which the pattern is similar to that of the autonomous response for the second-order subsystem. The behavior of the second-order subsystem depends on the pole location and the initial condition of the first-order subsystem. The differences between the autonomous response of the second-order subsystem and the response due to the exponentially decaying input are reported.

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