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CONCISENESS OF COPRIME COMMUTATORS IN FINITE GROUPS

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Abstract

Let G be a finite group. We show that the order of the subgroup generated by coprime γ_k-commutators (respectively δ_k-commutators) is bounded in terms of the size of the set of coprime γ_k-commutators (respectively δ_k-commutators). This is in parallel with the classical theorem due to Turner-Smith that the words γ_k and δ_k are concise.

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1. Introduction

Let F be the free group freely generated by $x_1, \dots, x_m$. Any nonidentity element of F is called a group-word in the variables $x_1, \dots, x_m$. Given a group-word, we think of it primarily as a function of m variables defined on any given group G. The verbal subgroup $w(G)$ of G determined by w is the subgroup generated by the set G_w consisting of all values $w(g_1, \dots, g_n)$, where $g_1, \dots, g_n$ are elements of G. A word w is said to be concise if whenever G_w is finite for a group G, it always follows that $w(G)$ is finite. More generally, a word w is said to be concise in a class of groups $\mathcal{X}$ if whenever G_w is finite for a group $G \in \mathcal{X}$, it always follows that $w(G)$ is finite. In the 1960s P. Hall asked whether every word is concise but later Ivanov proved that this problem has a negative solution in its general form [6] (see also [9, p. 439]). On the other hand, many important words are known to be concise. For instance, Turner-Smith [15] showed that the *lower central words* γ_k and the *derived words* δ_k are concise; here the words γ_k and δ_k are defined by the positions $\gamma_1 = \delta_0 = x_1$, $\gamma_{k+1} = [\gamma_k, x_{k+1}]$ and $\delta_{k+1} = [\delta_k, \delta_k]$. Wilson showed in [16] that the multilinear commutator words (outer commutator words) are concise. It was proved by Merzlyakov [8] that every word is concise in the class of linear groups.

In [3] a word w was called boundedly concise in a class of groups $\mathcal{X}$ if for every integer m there exists a number $\nu = \nu(\mathcal{X}, w, m)$ such that whenever $|G_w| \le m$ for a group $G \in \mathcal{X}$ it always follows that $|w(G)| \le \nu$. Fernández-Alcober and Morigi

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[4] showed that every word which is concise in the class of all groups is boundedly concise. Moreover they showed that whenever w is a multilinear commutator word having at most m values in a group G, one has $|w(G)| \leq (m-1)^{(m-1)}$. Questions on conciseness of words in the class of residually finite groups have been tackled in [1]. It was shown that if w is a multilinear commutator word and q a prime-power, then the word w^q is concise in the class of residually finite groups; and if $w = \gamma_k$ is the kth lower central word and q a prime-power, then the word w^q is boundedly concise in the class of residually finite groups.

The concept of (bounded) conciseness can be applied in a much wider context. Suppose $\mathcal{X}$ is a class of groups and $\phi(G)$ is a subset of G for every group $G \in \mathcal{X}$. One can ask whether the subgroup generated by $\phi(G)$ is finite whenever $\phi(G)$ is finite. In the present paper we show bounded conciseness of coprime commutators in finite groups.

The coprime commutators γ_k^* and δ_k^* were introduced in [13] as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. Let G be a finite group. Every element of G is both a γ_1^*-commutator and a δ_0^*-commutator. Now let $k \geq 2$ and let X be the set of all elements of G that are powers of γ_{k-1}^*-commutators. An element g is a γ_k^*-commutator if there exist $a \in X$ and $b \in G$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. For $k \geq 1$ let Y be the set of all elements of G that are powers of δ_{k-1}^*-commutators. The element g is a δ_k^*-commutator if there exist $a, b \in Y$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. The subgroups of G generated by all γ_k^*-commutators and all δ_k^*-commutators will be denoted by $\gamma_k^*(G)$ and $\delta_k^*(G)$, respectively. One can easily see that if N is a normal subgroup of G and x an element whose image in G/N is a γ_k^*-commutator (respectively a δ_k^*-commutator), then there exists a γ_k^*-commutator $y \in G$ (respectively a δ_k^*-commutator) such that $x \in yN$. It was shown in [13] that $\gamma_k^*(G) = 1$ if and only if G is nilpotent and $\delta_k^*(G) = 1$ if and only if the Fitting height of G is at most k. It follows that for every $k \geq 2$ the subgroup $\gamma_k^*(G)$ is precisely the last term of the lower central series of G (which is sometimes denoted by $\gamma_\infty(G)$) while for every $k \geq 1$ the subgroup $\delta_k^*(G)$ is precisely the last term of the lower central series of $\delta_{k-1}^*(G)$. In the present paper we prove the following results.

Theorem 1.1. *Let $k \geq 1$ and G a finite group in which the set of γ_k^*-commutators has size m. Then $|\gamma_k^*(G)|$ is m-bounded.*

Theorem 1.2. *Let $k \geq 0$ and G a finite group in which the set of δ_k^*-commutators has size m. Then $|\delta_k^*(G)|$ is m-bounded.*

We remark that the bounds for $|\gamma_k^*(G)|$ and $|\delta_k^*(G)|$ in the above results do not depend on k. Thus, we observe here the phenomenon that in [4] was dubbed "uniform conciseness". We make no attempts to provide explicit bounds for $|\gamma_k^*(G)|$ and $|\delta_k^*(G)|$ in Theorems 1.1 and 1.2. Throughout the paper we use the term m-bounded to mean that the bound is a function of m.

2. Preliminaries

We begin with a well-known result about coprime actions on finite groups. Recall that $[K, H]$ is the subgroup generated by $\{[k, h] : k \in K, h \in H\}$, and $[K,_i H] = [[K,_{i-1} H], H]$ for $i \geq 2$.

Lemma 2.1 ([5], Lemma 4.29). *Let A act via automorphisms on G, where A and G are finite groups, and suppose that $(|G|, |A|) = 1$. Then $[G, A, A] = [G, A]$.*

For the following result from [14], recall that a subset B of a group A is *normal* if B is a union of conjugacy classes of A.

Lemma 2.2. *Let A be a group of automorphisms of a finite group G with $(|A|, |G|) = 1$. Suppose that B is a normal subset of A such that $A = \langle B \rangle$. Let $k \geq 1$ be an integer. Then $[G, A]$ is generated by the subgroups of the form $[G, b_1, \ldots, b_k]$, where $b_1, \ldots, b_k \in B$.*

The following is an elementary property of δ_k^*-commutators.

Lemma 2.3. *Let G be a finite group. For k a non-negative integer,*

$$\delta_k^*(\delta_1^*(G)) = \delta_{k+1}^*(G).$$

Proof. We argue by induction. For $k = 0$, the result is obvious by the definition of δ_0^*-commutators.

Suppose the result holds for $k - 1$. So

$$\delta_{k-1}^*(\delta_1^*(G)) = \delta_k^*(G).$$

It was mentioned in the introduction that $\delta_{k+1}^*(G) = \gamma_\infty(\delta_k^*(G))$. By induction,

$$\delta_{k+1}^*(G) = \gamma_\infty(\delta_{k-1}^*(\delta_1^*(G))),$$

and viewing $\delta_1^*(G)$ as the group in consideration,

$$\gamma_\infty(\delta_{k-1}^*(\delta_1^*(G))) = \delta_k^*(\delta_1^*(G))$$

as required. □

Here is a helpful observation that we will use in both of our main results. Recall that a Hall subgroup of a finite group is a subgroup whose order is coprime to its index. Also, a finite group G is metanilpotent if and only if $\gamma_\infty(G)$ is nilpotent.

Lemma 2.4. *Let G be a finite metanilpotent group and P a Sylow p-subgroup of $\gamma_\infty(G)$, and let H be a Hall p'-subgroup of G. Then $P = [P, H]$.*

Proof. For simplicity, we write K for $\gamma_\infty(G)$. By passing to the quotient $G/O_{p'}(G)$, we may assume that $P = K$.

Let P_1 be a Sylow p-subgroup of G. So $G = P_1 H$. Now P_1/P is normal in G/P as G/P is nilpotent, but also $P \leq P_1$; hence P_1 is normal in G. It follows that $K = [P_1, H]$, since in a nilpotent group all coprime elements commute. By Lemma 2.1, $[P_1, H, H] = [P_1, H] = P$, and so $P = [P_1, H] = [P, H]$. □

In the proofs of our main results we often reduce to the following case.

Lemma 2.5. *Let i and m be positive integers. Let P be an abelian p-group acted on by a p'-group A such that*

$$|\{[x, a_1, \ldots, a_i] : x \in P, a_1, \ldots, a_i \in A\}| = m.$$

Then $|[P,_i A]| = 2^m$, so is m-bounded.

Proof. We enumerate the set $\{[x, a_1, \ldots, a_i] : x \in P, a_1, \ldots, a_i \in A\}$ as $\{c_1, \ldots, c_m\}$. As P is abelian,

$$[x, a_1, \ldots, a_i]^l = [x^l, a_1, \ldots, a_i] \qquad (\dagger)$$

for all $x \in P, a_1, \ldots, a_i \in A$, and l a positive integer.

Consider $g \in [P,_i A]$, which can be expressed as some product $c_1^{l_1} \ldots c_m^{l_m}$ for non-negative integers $l_1, \ldots, l_m$. We claim that $l_1, \ldots, l_m \in \{0, 1\}$. For, if $l_j > 1$ with $j \in \{1, \ldots, m\}$, we know from (†) that $c_j^{l_j} \in \{c_1, \ldots, c_m\}$. We replace all such $c_j^{l_j}$ accordingly, so that g is now expressed as $c_1^{k_1} \ldots c_m^{k_m}$ with $k_1, \ldots, k_m \in \{0, 1\}$. Hence $|[P,_i A]| = 2^m$. □

The well-known Focal Subgroup Theorem [12, 10.34 Corollary, p. 255] states that if G is a finite group and P a Sylow p-subgroup of G, then $P \cap G'$ is generated by the set of commutators $\{[g, z] \mid g \in G,\ z \in P,\ [g, z] \in P\}$. In particular, it follows that $P \cap G'$ can be generated by commutators lying in P. This observation led to the question on generation of Sylow subgroups of verbal subgroups of finite groups. More specifically, the following problem was addressed in [2].

Given a multilinear commutator word w and a Sylow p-subgroup P of a finite group G, is it true that $P \cap w(G)$ can be generated by w-values lying in P?

The answer to this is still unknown. The main result of [2] is that if G has order $p^a n$, where n is not divisible by p, then $P \cap w(G)$ is generated by nth powers of w-values. In the present paper we will require a result on generation of Sylow subgroups of $\delta_k^*(G)$.

Lemma 2.6. *Let $k \geq 0$ and let G be a finite soluble group of order $p^a n$, where p is a prime and n is not divisible by p, and let P be a Sylow p-subgroup of G. Then $P \cap \delta_k^*(G)$ is generated by nth powers of δ_k^*-commutators lying in P.*

It seems likely that Lemma 2.6 actually holds for all finite groups. In particular, the result in [2] was proved without the assumption that G is soluble. It seems though that proving Lemma 2.6 for arbitrary groups is a complicated task. Indeed, one of the tools used in [2] is the proof of the Ore Conjecture by Liebeck, O'Brien, Shalev, and Tiep [7] that every element of any finite simple group is a commutator. Recently it was conjectured in [13] that every element of a finite simple group is a commutator of elements of coprime orders. If this is confirmed, then extending Lemma 2.6 to arbitrary groups would be easy. However the conjecture that every element of a finite simple group is a commutator of elements of coprime orders is proved only for the alternating groups [13] and the groups $PSL(2, q)$ [10]. Thus, we prove Lemma 2.6 only for soluble groups, which is adequate for the purposes of the present paper.

Before we embark on the proof of Lemma 2.6, we note a key result from [2] that we will need.

Lemma 2.7. *Let G be a finite group, and let P be a Sylow p-subgroup of G. Assume that $N \leq L$ are two normal subgroups of G, and use bar notation in the quotient group G/N. Let X be a normal subset of G consisting of p-elements such that $\overline{P} \cap \overline{L} = \langle \overline{P} \cap \overline{X} \rangle$. Then $P \cap L = \langle P \cap X, P \cap N \rangle$.*

We are now ready to prove Lemma 2.6.

Proof. Let G be a counter-example of minimal order. Then $k \geq 1$.

By induction on the order of G, the lemma holds for every proper subgroup and every proper quotient of G. We observe that $\delta_1^*(G) < G$ since G is not perfect, and by Lemma 2.3, $\delta_{k+1}^*(G) = \delta_k^*(\delta_1^*(G))$. Since the result holds for $\delta_1^*(G)$, it follows that $P \cap \delta_{k+1}^*(G)$ is generated by nth powers of δ_k^*-commutators in G. Note that we made use of Remark 3.2 of [2].

If $\delta_{k+1}^*(G) \neq 1$, by induction the result holds for $G/\delta_{k+1}^*(G)$. Combining this with the fact that $P \cap \delta_{k+1}^*(G)$ can be generated by nth powers of δ_k^*-commutators, we get a contradiction by Lemma 2.7. Hence $\delta_{k+1}^*(G) = 1$. Further $O_{p'}(G) = 1$ since G is a minimal counter-example. Therefore $\delta_k^*(G) \subseteq P$, so $P \cap \delta_k^*(G)$ is generated by nth powers of δ_k^*-commutators lying in P. We have our required contradiction.

□

3. Proofs of the main results

We mention here a result of Schur and Wiegold. The much celebrated Schur Theorem states that if G is a group with $|G/Z(G)|$ finite, then $|G'|$ is finite. It is implicit in the work of Schur that if $|G/Z(G)| = m$, then $|G'|$ is m-bounded. However, Wiegold produced a shorter proof of this second statement, which also gives the best possible bound. See Robinson ([11], pages 102-103) for details.

For the proof of Theorem 1.2, we require the following result from [13].

Lemma 3.1. *Let G be a finite group and let $y_1, \dots, y_k$ be δ_k^*-commutators in G. Suppose $y_1, \dots, y_k$ normalize a subgroup N such that $(|y_i|, |N|) = 1$ for every $i = 1, \dots, k$. Then for every $x \in N$ the element $[x, y_1, \dots, y_k]$ is a δ_{k+1}^*-commutator.*

Now we are ready to begin.

Proof of Theorem 1.1. Let X be the set of all γ_k^*-commutators. We wish to show that if $|X| = m$, then $|\gamma_k^*(G)|$ is m-bounded. For convenience we write K for $\langle X \rangle$. Of course, $K = \gamma_\infty(G)$.

The subgroup $C_G(X)$ has index $\leq m!$, so $|K/Z(K)| \leq m!$ too. By Schur, K' has m-bounded order. Therefore, by passing to the quotient, we may assume $K' = 1$, and so K is abelian with G metanilpotent.

It is enough to bound the order of each Sylow subgroup of K. We choose a Sylow p-subgroup P. By passing to the quotient $G/O_{p'}(G)$, we may assume $K = P$.

By Lemma 2.4, a Hall p'-subgroup H of G satisfies $P = [P,_{k-1} H]$. We know that P is abelian and P is normal in PH.

We denote the set $\{[x, h_1, \dots, h_{k-1}] : x \in P, h_1, \dots, h_{k-1} \in H\}$ by $\hat{X}$.

For $x \in P$ and $h_1, \dots, h_{i-1} \in H$, where $i \geq 2$, we note that $[x, h_1, \dots, h_{i-1}]$ is a γ_i^*-commutator. Therefore $\hat{X} \subseteq X$, and $|\hat{X}| \leq m$.

By Lemma 2.5, it follows that $|[P,_{k-1} H]|$ is m-bounded. Appealing to Lemma 2.4, we conclude that $|P|$ is m-bounded. □

Proof of Theorem 1.2. Let X be the set of δ_k^*-commutators in G. We wish to show that if $|X| = m$, then $|\delta_k^*(G)|$ is m-bounded. We recall that $\delta_k^*(G) = \gamma_\infty(\delta_{k-1}^*(G))$. For ease of notation we define $Q := \delta_{k-1}^*(G)$, and we write K for $\delta_k^*(G)$.

The subgroup $C_G(X)$ has index $\leq m!$ in G, so $|K/Z(K)| \leq m!$ and as in the proof of Theorem 1.1, we may assume $K' = 1$. Hence K is assumed to be abelian with Q metanilpotent. In what follows, we now restrict to the group Q.

It is sufficient to show that the order of each Sylow subgroup of K is m-bounded. We choose P a Sylow p-subgroup of K. By passing to the quotient $G/O_{p'}(G)$, we may assume $K = P$.

By Lemma 2.4, a Hall p'-subgroup H of Q satisfies $P = [P, H]$. By Lemma 2.6, since H is generated by its Sylow subgroups, we have H is generated by a normal subset B of powers of δ_{k-1}^*-commutators that are of p' order.

Lemma 2.2 now implies that $[P, H]$ is generated by subgroups $[P, b_1, \dots, b_k]$ for $b_1, \dots, b_k \in B$. By Lemma 3.1, if $x \in P$, then $[x, b_1, \dots, b_k]$ is a δ_k^*-commutator, and we deduce that $|[P, b_1, \dots, b_k]|$ is m-bounded.

It follows that the number of generators of $[P, H]$ is at most m, and futhermore the exponent of $[P, H]$ is m-bounded. Hence, the finite abelian p-group $P = [P, H]$ has m-bounded order. □

References

[1] C. Acciarri, P. Shumyatsky, On words that are concise in residually finite groups, submitted. arXiv:1212.0581[math.GR].

[2] C. Acciarri, G. A. Fernández-Alcober, P. Shumyatsky, A focal subgroup theorem for outer commutator words, *J. Group Theory* **15** (2012), 397–405.

[3] S. Brazil, A. Krasilnikov, P. Shumyatsky, Groups with bounded verbal conjugacy classes, *J. Group Theory* **9** (2006), 127–137.

[4] G. A. Fernández-Alcober, M. Morigi, Outer commutator words are uniformly concise. *J. London Math. Soc.* **82** (2010), 581–595.

[5] I. M. Isaacs, *Finite Group Theory*, Amer. Math. Soc., vol. 92, 2008.

[6] S. V. Ivanov, P. Hall's conjecture on the finiteness of verbal subgroups, *Izv. Vyssh. Ucheb. Zaved.* **325** (1989), 60–70.

[7] M. W. Liebeck, E. A. O' Brien, A. Shalev, P. H. Tiep, The Ore conjecture, *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 4, 939–1008.

[8] Ju. I. Merzlyakov, Verbal and marginal subgroups of linear groups, *Dokl. Akad. Nauk SSSR* **177** (1967), 1008–1011.

[9] A. Yu. Ol'shanskii, *Geometry of Defining Relations in Groups*, Mathematics and its applications **70** (Soviet Series), Kluwer Academic Publishers, Dordrecht, 1991.
[10] M. A. Pellegrini, P. Shumyatsky, Coprime commutators in $\mathrm{PSL}(2,q)$, *Arch. Math.* **99** (2012), 501–507.
[11] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Part 1, Springer-Verlag, 1972.
[12] J. S. Rose, *A Course on Group Theory*, Dover Publications, New York, 1994.
[13] P. Shumyatsky, Commutators of elements of coprime orders in finite groups, *Forum Mathematicum*, doi:10.1515/forum-2012-0127, to appear.
[14] P. Shumyatsky, On the exponent of a verbal subgroup in a finite group, *J. Austral. Math. Soc.*, doi:10.1017/S1446788712000341, to appear.
[15] R. F. Turner-Smith, Finiteness conditions for verbal subgroups, *J. London Math. Soc.* **41** (1966), 166–176.
[16] J. Wilson, On outer-commutator words, *Can. J. Math.* **26** (1974), 608–620.

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