GROUPS OF p-DEFICIENCY ONE

ANITHA THILLAISUNDARAM

Abstract. In a previous paper, Button and I proved that all finitely presented groups of p-deficiency greater than one are p-large. Here I prove that groups with a finite presentation of p-deficiency one possess a finite index subgroup that surjects onto the integers. This implies that these groups do not have Kazhdan's property (T). Additionally, I prove that the aforementioned result of Button and myself implies a result of Lackenby.

Introduction

This paper continues from my joint paper with Button [5].

Throughout this paper, p denotes a prime. We recall the following.

Definition. [12, 13] Let G be a finitely generated group. Say $G \cong \langle X|R\rangle$, with $|X|$ finite. For a prime p, the *p-deficiency* of G with presentation $\langle X|R\rangle$ is

$$\mathrm{def}_p(G; X, R) = |X| - \sum_{r\in R} p^{-\nu_p(r)},$$

where $\nu_p(r) = \max\left\{k \geq 0 \mid \exists w \in F(X),\ w^{p^k} = r\right\}$. The p-deficiency of G is then defined to be the supremum of $\mathrm{def}_p(G; X, R)$ over all presentations $\langle X|R\rangle$ of G with $|X|$ finite.

This is similar to an older concept:

Definition. The *deficiency* of a group G is

$$\mathrm{def}(G) = \sup_{\langle X|R\rangle} \{|X| - |R| : G \cong \langle X|R\rangle\}.$$

Recent and interesting developments in p-deficiency (and in particular, on groups of p-deficiency one) were made by Barnea & Schlage-Puchta [2]. The reader should note our different definition of p-deficiency, which is larger by 1 compared to Schlage-Puchta's definition [13].

We recall the concepts of largeness and p-largeness.

Definition. [10] Let G be a group, and let p be a prime. Then

• G is *large* if some (not necessarily normal) subgroup with finite index admits a non-abelian free quotient;

• G is *p-large* if some normal subgroup with index a power of p admits a non-abelian free quotient.

Date: 8th November 2012.

The main result of [5] is the following.

Theorem 1. *([5], Theorem 2.2) Let p be a prime. If G is a finitely presented group with p-deficiency greater than one, then G is p-large.*

Theorem 1 is proved using results of Lackenby ([10], Theorem 1.15) and Schlage-Puchta [12, 13]. See [5] for details of the proof.

By Corollary 2.1 of [5], groups with a finite presentation of p-deficiency one are infinite. In this paper, we prove the following.

Theorem. *Let Γ be a finitely presented group with a presentation of p-deficiency equal to one, for some prime p. Then Γ has a finite index subgroup H that surjects onto $\mathbb{Z}$.*

The Related Burnside Problem ([8], Problem 8.52) asks whether or not there exist infinite finitely presented torsion groups. The theorem above extends Corollary 2.4 of [5] to give the following response to the Related Burnside Problem.

Corollary. *Let G be an infinite finitely presented group with a presentation of p-deficiency greater than or equal to one, for some prime p. Then G is not torsion.*

We note here the definition of Kazhdan's property (T). See [3] for more information on property (T).

Definition. [3] Let Γ be a finitely generated group.

(a) Given a unitary representation V of Γ and a generating set S of Γ, we define $\kappa(\Gamma; S; V)$ to be the largest $\varepsilon \geq 0$ such that for any $v \in V$ there exists $s \in S$ with $||sv - v|| \geq \varepsilon ||v||$.

(b) Given a generating set S of Γ, the *Kazhdan constant* $\kappa(\Gamma; S)$ is defined to be the infimum of the set $\{\kappa(\Gamma; S; V)\}$ where V runs over all unitary representations of Γ without non-zero invariant vectors.

(c) The group Γ is called a *Kazhdan group* (equivalently Γ is said to have *Kazhdan's property (T)*) if $\kappa(\Gamma; S) > 0$ for some (hence any) finite generating set S of Γ.

It is proved in [3] (Corollaries 1.3.6 & 1.7.2) that if a group has property (T), then its finite index subgroups must have finite abelianization.

Lastly, we include the definition of linear growth of mod p homology for later use.

Definition. [10] We say that a collection of finite index subgroups $\{G_i\}$ has *linear growth of mod p homology* if

$$\inf_i \frac{d_p(G_i)}{[G : G_i]} > 0.$$

Section 1 of this paper presents the proof of our main result. The corollaries of our main result are the content of Section 2. We finish with Section 3 which gives an interesting example of a group with a finite presentation of 3-deficiency one, and we comment on Ershov's finitely presented Golod-Shafarevich group with property (T).

This paper is mostly an extract, which was under the supervision of Jack Button, of my PhD thesis.

I would like to thank Yiftach Barnea, Alex Bartel, Mikhail Ershov and the referee for their helpful comments on this paper. Also I further thank the referee for providing a clearer proof of the main result.

And I am grateful to the Cambridge Commonwealth Trust, the Cambridge Overseas Research Scholarship, and the Leslie Wilson Scholarship (from Magdalene College, Cambridge) for their financial support.

1. Main Result

There is an amount of ambiguity in saying that a group is of p-deficiency one. Formally, the p-deficiency of a group is the supremum of $\mathrm{def}_p(\langle X|R\rangle)$ over all presentations $\langle X|R\rangle$ of the group. Therefore it is theoretically possible for the p-deficiency of a group to be one in the limit, but with none of $\mathrm{def}_p(\langle X|R\rangle)$ being equal to one.

We avoid this delicate situation by insisting that the group has a presentation of p-deficiency one. This is the convention that we adopt whenever we deal with p-deficiency one groups.

First, we note a result from [1].

Theorem 2. [1] *Let G be a group with presentation*

$$\langle a_1, \ldots, a_n | 1 = w_1^{r_1} = \ldots = w_m^{r_m} \rangle$$

where each w_j is a word in the a_i and their inverses. Suppose that H is a normal subgroup of G of index $N < \infty$ and that for each j, $w_j^k \notin H$ for $k = 1, \ldots, r_j - 1$. Then the rank of the abelianization of H is at least

$$1 + N \left(n - 1 - \sum_i \frac{1}{r_i} \right).$$

We now prove the following.

Theorem 3. *Let Γ be a finitely presented group with a presentation of p-deficiency equal to one, for some prime p. Then Γ has a finite index subgroup H that surjects onto $\mathbb{Z}$.*

Proof. Let Γ be a finitely presented group with a presentation of p-deficiency equal to one. So we have

$$\Gamma \cong \langle x_1, \ldots, x_d | w_1, \ldots, w_r, w_{r+1}^{p^{a_{r+1}}}, \ldots, w_q^{p^{a_q}} \rangle$$

with

$$\mathrm{def}_p(\Gamma) = d - r - \sum_{i=r+1}^{q} \frac{1}{p^{a_i}} = 1$$

where $d, q \in \mathbb{N}$, $0 \leq r \leq q$, and $a_{r+1} \leq \ldots \leq a_q$ are positive integers.

For $i \in \{r + 1, \ldots, q\}$, we say that w_i has *exact order* if there is some normal subgroup H of finite index in Γ such that $w_i^{p^{a_i - 1}} \notin H$. That is, w_i has order p^{a_i} as in the presentation above.

Consider w_q. Either w_q has exact order in some finite index normal subgroup or it does not.

If it does not, consider now

$$G = \langle x_1, \ldots, x_d | w_1, \ldots, w_r, w_{r+1}^{p^{a_{r+1}}}, \ldots, w_{q-1}^{p^{a_{q-1}}}, w_q^{p^{a_q + 1}} \rangle.$$

Note that

$$\Gamma \cong G/\langle\langle w_q^{p^{a_q}} \rangle\rangle$$

and as $\mathrm{def}_p(G) > 1$ we have that G is p-large.

Recall the definition of p-large. Let H be a normal subgroup in G of index p^k, for $k \geq 0$, such that there exists a surjection $\psi : H \twoheadrightarrow F_2$.

For ease of notation, we write $w := w_q$ and $a := a_q$. We may assume that the order of w in G is p^{a+1}, as if $o(w) < p^{a+1}$, then $\Gamma = G$ is p-large, and we are done.

Now, our plan is to consider $G/\langle\langle w^{p^a} \rangle\rangle$ and show that this quotient group has a finite index subgroup that surjects onto $\mathbb{Z}$. As $\Gamma \cong G/\langle\langle w^{p^a} \rangle\rangle$, this completes the proof of our theorem.

Consider the order of $\overline{w}$, the image of w in G/H.

a) If $o(\overline{w})$ in G/H is $< p^a$, then this implies that $w^{p^{a-1}} \in H$. We will show that $G/\langle\langle w^{p^a} \rangle\rangle$ is p-large to obtain our result.

Let $k_1, \ldots, k_s$ be a set of representatives for the cosets of H in G. Let m $(\leq p^{a-1})$ be the smallest positive integer such that $k_j w^m k_j^{-1} \in H$, for each j. Note that m divides $p^k = [G : H]$. Let n be any positive integer, and let $G_{mn} = \langle\langle w^{mn} \rangle\rangle$ be the subgroup of G generated normally by w^{mn}. Note that this is contained in H, and is in fact the subgroup of H normally generated by $\{k_j w^{mn} k_j^{-1} : 1 \leq j \leq s\}$. Now $\{\psi(k_j w^{mn} k_j^{-1}) : 1 \leq j \leq s\}$ is a collection of elements in F_2. The key thing to note here, is that $k_j w^{mn} k_j^{-1}$ all have orders a power of p in H, and so their images under ψ must be trivial in F_2. So $G_{mn} \leq \ker \psi$, and we have the induced surjection $\overline{\psi} : H/G_{mn} \twoheadrightarrow F_2$. Now H/G_{mn} has finite p^{th} power index in G/G_{mn}. Therefore G/G_{mn} is p-large. Finally, we take $mn = p^a$, and therefore $G/\langle\langle w^{p^a} \rangle\rangle \cong \Gamma$ is p-large. The result now follows for Γ.

b) If $o(\overline{w})$ in G/H is $\geq p^a$, then in $G/H\langle\langle w^{p^a} \rangle\rangle$ the image of w has order dividing p^a. As this is a finite p-quotient of Γ, we use the fact that w has exact order p^a in $G/H\langle\langle w^{p^a} \rangle\rangle$, to deduce that w has exact order p^a in Γ. As we are assuming here that w does not have exact order in Γ, this case (b) is not possible.

So we see from the above that if w_q does not have exact order, then Γ is p-large, and the statement of the theorem is true.

Now we assume that w_q has exact order with respect to some finite index normal subgroup H_q. Henceforth we only consider normal subgroups of Γ that are contained in H_q.

Consider w_{q-1}. Either w_{q-1} has exact order with respect to some finite index normal subgroup contained in H_q, or it does not. If w_{q-1} does not have exact order, then similar to the above arguments, we have that Γ is p-large.

If w_{q-1} has exact order with respect to some finite index normal subgroup H_{q-1} contained in H_q, then we henceforth only consider normal subgroups of Γ that are contained in H_{q-1}.

And so on. Either Γ is proved to be p-large at some stage, or we end up with some finite index normal subgroup H_{r+1} such that w_k has exact order with respect to H_{r+1} for all $k = r+1, \ldots, q$. Then the rank of the abelianization of H_{r+1} is at least one by Theorem 2. Hence H_{r+1} surjects onto $\mathbb{Z}$, as required. □

The second line of part (b) draws on the following simple fact from finite p-groups.

Lemma. *Let g be an element of a finite p-group G, and say $o(g) = p^k$ for $k > 0$. Let $N = \langle\langle g^{p^{k-1}} \rangle\rangle$. Then $g^{p^{k-2}} \notin N$.*

Proof. We consider the Frattini subgroup $\Phi(N)$ of N, which is defined to be the intersection of all maximal subgroups of N. It is well-known that $\Phi(N)$ is characteristic in N, and that $\Phi(N) = N^p N'$ as N is a finite p-group. As $\Phi(N)$ is characteristic in N, and N is normal in G, we have that $\Phi(N)$ is normal in G.

Firstly, we note that we cannot have $g^{p^{k-1}}$ belonging to $\Phi(N)$: if $g^{p^{k-1}} \in \Phi(N)$, then all conjugates $h^{-1} g^{p^{k-1}} h$, for $h \in G$, also lie in $\Phi(N)$. This means that $N = \Phi(N)$, which is impossible.

Now suppose that $g^{p^{k-2}} \in N$. Then

$$(g^{p^{k-2}})^p = g^{p^{k-1}} \in N^p \leq \Phi(N),$$

a contradiction. Thus $g^{p^{k-2}} \notin N$, as required. □

Remark 4. The referee has provided a very nice alternative for part (a) of the proof of Theorem 3, as seen here.

Suppose that the w_i's do not have exact order. Then for every finite index normal subgroup H of Γ there exists some k such that $w_k^{p^{a_k - 1}} \in H$. So under the application of the Reidemeister-Schreier rewriting process, the relators involving w_k remain p^{th} powers in the presentation for H. Thus, on computing the rank of the mod p homology of H (i.e. $d_p(H)$), we may disregard contributions from these relators involving w_k.

Applying this to the derived p-series of Γ (i.e., $\Gamma^{(0)} = \Gamma$, $\Gamma^{(i)} = [\Gamma^{(i-1)}, \Gamma^{(i-1)}](\Gamma^{(i-1)})^p$ for $i \in \mathbb{N}$), we deduce that the derived p-series of Γ has linear growth of mod p homology. Theorem 1.12 of [10] now implies that Γ is p-large.

2. Corollaries

Theorem 3 together with Corollary 2.4 of [5] imply the following response to the Related Burnside Problem.

Corollary 5. *Let G be an infinite finitely presented group with a presentation of p-deficiency greater than or equal to one, for some prime p. Then G is not torsion.*

The next corollary incorporates Kazhdan's property (T).

Corollary 6. *Let G be an infinite finitely presented group with a presentation of p-deficiency greater than or equal to one, for some prime p. Then G does not have property (T).*

Proof. By Theorems 1 and 3, we know that G has a finite index subgroup H such that H surjects onto $\mathbb{Z}$. The result now follows from Corollary 1.3.6 and Corollary 1.7.2 of [3]. □

Next, we have the following result from [9].

Theorem 7. [9] *Let G be a finitely generated, large group and let $g_1, \ldots, g_r$ be a collection of elements of G. Then for infinitely many integers n, $G/\langle\langle g_1^n, \ldots, g_r^n \rangle\rangle$ is also large. Indeed, this is true when n is any sufficiently large multiple of $[G : H]$, where H is any finite index normal subgroup of G that admits a surjective homomorphism onto a non-abelian free group.*

Part (a) of the proof of Theorem 3 follows the proof of Theorem 7 closely. Below is a stronger statement for free groups which is used in the proof of Theorem 7.

Theorem 8. [9] *Let F be a finitely generated, non-abelian free group. Let $g_1, \ldots, g_r$ be a collection of elements of F. Then, for all but finitely many integers n, the quotient $F/\langle\langle g_1^n, \ldots, g_r^n \rangle\rangle$ is large.*

The above theorem has a topological proof. As in the proof of the Nielsen-Schreier Theorem on subgroups of free groups, F here is viewed as the fundamental group of a bouquet of circles. Then the quotient $F/\langle\langle g_1^n, \ldots, g_r^n \rangle\rangle$ is obtained by attaching 2-cells representing $g_1^n, \ldots, g_r^n$ along the circles. More details are to be found in [9].

Olshanskii and Osin give a shorter algebraic proof of Theorem 7 in [11]. The main body of Olshanskii and Osin's proof relies on Theorem 8, which they also prove with alternative algebraic arguments.

We remark here that Theorem 8 (and hence Theorem 7) follows from Theorem 1. We remind the reader that Theorem 1 relies on another result of Lackenby ([10], Theorem 1.15).

Corollary 9. *Let F be a free group of rank $r \geq 2$, with $g_1, \ldots, g_k$ arbitrary elements of F. Then $\overline{F} \cong F/\langle\langle g_1^q, \ldots, g_k^q \rangle\rangle$ is large for all but finitely many $q \in \mathbb{N}$.*

Proof. We consider the p-deficiency of $\overline{F}$:

$$\operatorname{def}_p(\overline{F}) \geq r - \frac{k}{p^{l_p}},$$

where p is some prime factor of q, and l_p is the highest power of p dividing q. By Theorem 1, the group $\overline{F}$ is p-large if $\operatorname{def}_p(\overline{F}) > 1$, that is, when $p^{l_p} > \frac{k}{r-1}$.

So as long as $p^{l_p} > \frac{k}{r-1}$ for at least one p dividing q, then $\overline{F}$ is large. That is, for all but finitely many $q \in \mathbb{N}$, the group $\overline{F}$ is large. □

Lackenby's proof of Theorem 8 (or Corollary 9) relies on topological arguments, which span over a few pages. Here, Theorem 1 has enabled us to present a short proof of a different spirit.

3. Examples

Clearly finitely presented groups of p-deficiency one exist and examples include the infinite dihedral group $D_\infty = \langle x_1, x_2 | x_1^2, x_2^2 \rangle$, all groups of deficiency one, and the group $P = \langle x, y, z | x^3, y^3, z^3, (xy)^3, (xz)^3, (yz)^3 \rangle$. The group D_∞ is not torsion nor large. The groups of deficiency one are not torsion but some are large (see [4]). The group P was verified by MAGMA to be 3-large (and hence is not torsion). For the group P, we used the approach shown below, as is similar to Subsection 4.2 of [5].

Claim. The group $P = \langle x, y, z | x^3, y^3, z^3, (xy)^3, (xz)^3, (yz)^3 \rangle$ is 3-large.

Proof. Using MAGMA's LowIndexNormalSubgroups function, we considered the following index three normal subgroup of P:

$$C = \langle a, b, c, d | [c^{-1}, a^{-1}], [d^{-1}, b^{-1}], adc^{-1}a^{-1}bcd^{-1}b^{-1} \rangle,$$

which was (at the time of writing) seventh on the list of fourteen normal subgroups with index at most three in P. The above presentation for C was obtained using MAGMA's Simplify function.

Then we formed the quotient

$$C/\langle\langle c, d \rangle\rangle$$

and we noticed that the quotient is isomorphic to $\langle a, b \rangle \cong F_2$. Hence C is 3-large by definition, and since C is normal in P of index 3, we have proved that P is 3-large, as required. □

With reference to Corollary 6, the following example is due to Ershov & Jaikin-Zapirain ([7], Proposition 7.4). Let $d \geq 6$ and $p > (d-1)^2$, then the group

$$G \cong \langle x_1, \ldots, x_d | \, [[x_i, x_j], x_i] = 1 \ \forall i \neq j, \ x_i^p = 1 \rangle$$

is a finitely presented Golod-Shafarevich group with property (T) (see [6] for further information). Naturally the p-deficiency of G is not one.

References

[1] D. Allcock, Spotting Infinite Groups, Math. Proc. Cambridge Philos. Soc. 125 (1999), no. 1, 39-42.
[2] Y. Barnea & J.-C. Schlage-Puchta, On p-deficiency in groups, arXiv:1106.3255v1.
[3] B. Bekka, P. de la Harpe & A. Valette, Kazhdan's Property (T), New Math. Monogr. 11, Cambridge Univ. Press, Cambridge, 2008.
[4] J. O. Button, Large groups of deficiency 1, Israel J. Math 167 (2008), 111-140.
[5] J. O. Button & A. Thillaisundaram, Applications of p-deficiency and p-largeness, Internat. J. of Algebra Comput., vol. 21, no.4 (2011), 547-574.
[6] M. Ershov, Golod-Shafarevich groups with property (T) and Kac-Moody groups, Duke Math. J. 145, no. 2 (2008), 309-339.
[7] M. Ershov & A. Jaikin-Zapirain, Property (T) for noncommutative universal lattices, Invent. Math. 179, no. 2 (2010), 303-347.
[8] E. I. Khukhro & V. D. Mazurov, Unsolved problems in group theory: the Kourovka notebook, Russian Academy of Sciences, 1995.
[9] M. Lackenby, Adding high powered relations to large groups, Math. Res. Lett. 14, no.6 (2007), 983-993.

[10] M. Lackenby, Detecting large groups, J. Algebra 324, no. 10 (2010), 2636-2657.
[11] A. Yu. Olshanskii & D. V. Osin, Large groups and their periodic quotients, Proc. Amer. Math. Soc. 136. no.3 (2008), 753-759.
[12] J.-C. Schlage-Puchta, Profinite Groups Meeting, 12th June 2009.
[13] J.-C. Schlage-Puchta, A p-group with positive rank gradient, arXiv:1105.1631 (2011).

Magdalene College, Cambridge, CB3 0AG, United Kingdom.
E-mail address: anitha.t@cantab.net