# ON FINITE SOLUBLE GROUPS WITH ALMOST FIXED-POINT-FREE AUTOMORPHISMS OF NON-COPRIME ORDER

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Abstract. It is proved that if a finite $p$-soluble group $G$ admits an automorphism $\varphi$ of order $p^n$ having at most $m$ fixed points on every $\varphi$-invariant elementary abelian $p'$-section of $G$, then the $p$-length of $G$ is bounded above in terms of $p^n$ and $m$; if in addition the group $G$ is soluble, then the Fitting height of $G$ is bounded above in terms of $p^n$ and $m$. It is also proved that if a finite soluble group $G$ admits an automorphism $\psi$ of order $p^a q^b$ for some primes $p, q$, then the Fitting height of $G$ is bounded above in terms of $|\psi|$ and $|C_G(\psi)|$.

*to Yurii Leonidovich Ershov on the occasion of his 75-th birthday*

## 1. Introduction

Studying groups with "almost fixed-point-free" automorphisms means obtaining restrictions on the structure of groups depending on their automorphisms and certain restrictions imposed on the fixed-point subgroups. In this paper we consider questions of bounding the $p$-length and Fitting height of finite $p$-soluble and soluble groups admitting almost fixed-point-free automorphisms of non-coprime order.

Let $\varphi \in \operatorname{Aut} G$ be an automorphism of a finite group $G$. Studying the structure of the group $G$ depending on $\varphi$ and the fixed-point subgroup $C_G(\varphi)$ is one of the most important and fruitful avenues in finite group theory. The celebrated Brauer–Fowler theorem [1] (bounding the index of the soluble radical in terms of the order of $|C_G(\varphi)|$ when $|\varphi| = 2$) and Thompson's theorem [2] (giving the nilpotency of $G$ when $\varphi$ is of prime order and acts fixed-point-freely, that is, $C_G(\varphi) = 1$) lie in the foundations of the classification of finite simple groups. The classification was used for obtaining further results on solubility of $G$, or of a suitable "large" subgroup. For example, using the classification Hartley [3] generalized the Brauer–Fowler theorem to any order of $\varphi$: the group $G$ has a soluble subgroup of index bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$.

Now suppose that the group $G$ is soluble. Further information on the structure of $G$ is sought first of all in the form of bounds for the Fitting height (nilpotent length). A bound for the Fitting height naturally reduces further studies to the case of nilpotent groups with (almost) fixed-point-free automorphisms, for which, in turn, problems arise of bounding the derived length, or the nilpotency class of the group or of a suitable "large" subgroup. Such bounds for nilpotent groups so far have been obtained in the cases of $\varphi$ being of prime order or of order 4 in [4, 5, 6, 7, 8, 9]. In addition, definitive general results have been obtained in the study of almost fixed-point-free $p$-automorphisms of finite $p$-groups [10, 11, 12, 13, 14, 15].

On bounding the Fitting height, especially strong results have been obtained in the case of soluble groups of automorphisms $A \leqslant \operatorname{Aut} G$ of coprime order. Thompson [16] proved that if both groups $G$ and $A$ are soluble and have coprime orders, then the Fitting height

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of $G$ is bounded in terms of the Fitting height of $C_G(A)$ and the number $\alpha(A)$ of prime factors of $|A|$ with account for multiplicities. Later the bounds in Thompson's theorem were improved in numerous papers, with definitive results obtained by Turull [17] and Hartley and Isaacs [18] with linear bounds in terms of $\alpha(A)$ for the Fitting height of the group or of a "large subgroup".

The case of non-coprime orders of $G$ and $A \leqslant \operatorname{Aut} G$ is more difficult. Bell and Hartley [19] constructed examples showing that for any non-nilpotent finite group $A$ there are soluble groups $G$ of arbitrarily high Fitting height admitting $A$ as a fixed-point-free group of automorphisms. But if $A$ is nilpotent and $C_G(A) = 1$, then the Fitting height of $G$ is bounded in terms of $\alpha(A)$ by a special case of Dade's theorem [20]. Unlike the aforementioned "linear" results in the coprime case, the bound in Dade's theorem is exponential. Improving this bound to a linear one is a difficult problem; it was tackled in some special cases by Ercan and Güloğlu [21, 22, 23].

In the almost fixed-point-free situation, even for a cyclic group of automorphisms $\langle\varphi\rangle \leqslant \operatorname{Aut} G$ it is still an open problem to obtain a bound for the Fitting height of a finite soluble group $G$ in terms of $|\varphi|$ and $|C_G(\varphi)|$ (this question is equivalent to the one recorded by Belyaev in Kourovka Notebook [24] as Hartley's Problem 13.8(a)). Beyond the fixed-point-free case of Dade's theorem, so far the only cases where an affirmative solution is known are the cases of automorphisms of primary order $p^n$ (Hartley and Turau [25]) and of biprimary order $p^a q^b$ (which is discussed in the present paper).

Another generalization of fixed-point-free automorphisms in the non-coprime case is Thompson's problem on bounding the $p$-length of a finite $p$-soluble group $G$ admitting a $p$-group of automorphisms $P$ that acts fixed-point-freely on every $P$-invariant $p'$-section of $G$. Rae [26] and Hartley and Rae [27] solved this problem in the affirmative for $p \neq 2$, as well as for cyclic $P$ for any $p$. A special case of this problem is when a $p$-soluble group $G$ admits a so-called $p^n$-splitting automorphism $\varphi$, which means that $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p^n-1}} = 1$ for all $x \in G$ (this also implies $\varphi^{p^n} = 1$); then of course $\varphi$ automatically acts fixed-point-freely on $\varphi$-invariant $p'$-sections. This case was actually considered earlier by Kurzweil [28] who obtained bounds for the Fitting height of a soluble group $G$, and these bounds were improved to linear ones by Meixner [29]. If it is only known that $\varphi$ induces a $p^n$-splitting automorphism on a $\varphi$-invariant Sylow $p$-subgroup of $G$, then there is already a bound in terms of $n$ for the $p$-length of $G$: for $p \neq 2$ such a bound was obtained by Wilson [30], and for all primes $p$ in [31] even under a weaker assumption.

In this paper we consider the natural generalization of Thompson's problem for a $p$-soluble group $G$ admitting an automorphism $\varphi$ of order $p^n$ in which the condition that $\varphi$ acts fixed-point-freely on $\varphi$-invariant $p'$-sections is replaced by that $\varphi$ acts almost fixed-point-freely on these sections. It is actually sufficient to impose the restriction on the number of fixed points of $\varphi$ only on elementary abelian $\varphi$-invariant $p'$-sections.

**Theorem 1.1.** *If a finite $p$-soluble group $G$ admits an automorphism $\varphi$ of order $p^n$ such that $\varphi$ has at most $m$ fixed points on every $\varphi$-invariant elementary abelian $p'$-section of $G$, then the $p$-length of $G$ is bounded above in terms of $p^n$ and $m$.*

It would be interesting to obtain a bound of the $p$-length in terms of $n$ (or at least in terms of $p^n$) for some subgroup of index bounded in terms of $p^n$ and $m$.

**Remark 1.2.** There is a certain similarity with the situation for a $p^n$-splitting automorphism described above. Namely, if, for a $p$-soluble group $G$ with an automorphism $\varphi$ of order $p^n$, instead of a restriction on the number of fixed points on $p'$-sections, we have a restriction $|C_P(\varphi)| = p^m$ on the number of fixed points of $\varphi$ in a $\varphi$-invariant Sylow $p$-subgroup $P$, then we also obtain a bound for the $p$-length of $G$. Indeed, then the derived length of $P$ is bounded in terms of $p$, $n$, and $m$ by Shalev's theorem [12], so the bound for the $p$-length immediately follows from the Hall–Higman theorems [32] for $p \neq 2$, and the theorems of Hoare [33], Berger and Gross [34], and Bryukhanova [35]. Moreover, by [13] the group $P$ even has a (normal) subgroup of index bounded in terms of $p$, $n$, and $m$ that has $p^n$-bounded derived length. Therefore by the Hall–Higman–Hartley Theorem 2.3 (see below) there is a characteristic subgroup $H$ of $G$ such that the $p$-length of $H$ is $p^n$-bounded and a Sylow $p$-subgroup of the quotient $G/H$ has order bounded in terms of $p$, $n$, and $m$.

For soluble groups, Theorem 1.1 can be combined with known results to give a bound for the Fitting height.

**Corollary 1.3.** *If a finite soluble group $G$ admits an automorphism $\varphi$ of order $p^n$ such that $\varphi$ has at most $m$ fixed points on every $\varphi$-invariant elementary abelian $p'$-section of $G$, then the Fitting height of $G$ is bounded above in terms of $p^n$ and $m$.*

The technique used in the proof of Theorem 1.1 is also applied in the proof of the soluble case of the following theorem on almost fixed-point-free automorphism of biprimary order; the reduction to the soluble case is given by Hartley' theorem [3] (based on the classification of finite simple groups).

**Theorem 1.4.** *If a finite group $G$ admits an automorphism $\varphi$ of order $p^a q^b$ for some primes $p, q$ and nonnegative integers $a, b$, then $G$ has a soluble subgroup whose index and Fitting height are bounded above in terms of $p^a q^b$ and $|C_G(\varphi)|$.*

Standard inverse limit arguments yield the following corollary for locally finite groups.

**Corollary 1.5.** *If a locally finite group $G$ contains an element $g$ of order $p^a q^b$ for some primes $p, q$ and nonnegative integers $a, b$ with finite centralizer $C_G(g)$, then $G$ has a subgroup of finite index that has a finite normal series with locally nilpotent factors.*

Another corollary is of more technical nature but it may be useful in further studies.

**Corollary 1.6.** *If a finite group $G$ admits an automorphism $\varphi$ such that there are at most two primes dividing both $|\varphi|$ and $|G|$, then $G$ has a soluble subgroup whose index and Fitting height are bounded above in terms of $|\varphi|$ and $|C_G(\varphi)|$.*

**Remark 1.7.** After this paper was prepared for publication, the author became aware of an unpublished manuscript of Brian Hartley, which contains the result of Theorem 1.4; the author together with A. Borovik and P. Shumyatsky published this manuscript as [36] on the web-site of the University of Manchester.

## 2. Preliminaries

Induced automorphisms of invariant sections are denoted by the same letters. The following lemma is well known.

**Lemma 2.1.** *If $\varphi$ is an automorphism of a finite group $G$ and $N$ is a normal $\varphi$-invariant subgroup, then $|C_{G/N}(\varphi)| \leqslant |C_G(\varphi)|$.*

The next lemma is also a well-known consequence of considering the Jordan normal form of a linear transformation of order $p^k$ in characteristic $p$.

**Lemma 2.2.** *If an elementary abelian $p$-group $P$ admits an automorphism $\varphi$ of order $p^k$ such that $|C_P(\varphi)| = p^m$, then the rank of $P$ is bounded in terms of $p^k$ and $m$.*

We shall use the following consequence of the Hall–Higman–type theorems in Hartley's paper [37].

**Theorem 2.3** (Hall–Higman–Hartley). *Let $P$ be a Sylow $p$-subgroup of a $p$-soluble group $G$. If $R$ is a normal subgroup of $P$ and the derived length of $R$ is $d$, then $R \leqslant O_{p',p,p',\dots,p',p}(G)$, where $p$ occurs on the right-hand side $d$ times if $p > 3$, $2d$ times if $p = 3$, and $3d$ times if $p = 2$.*

*Proof.* As a refinement of some of the Hall–Higman theorems [32], Hartley [37] proved that if $A$ is an abelian normal subgroup of a Sylow $p$-subgroup of $G$, then

$$A \leqslant O_{p',p}(G) \qquad \text{if} \quad p > 3,$$

$$A \leqslant O_{3',3,3',3}(G) \qquad \text{if} \quad p = 3,$$

and

$$A \leqslant O_{2',2,2',2,2'2}(G) \qquad \text{if} \quad p = 2.$$

The result follows from these inclusions for $A = R^{(d-1)}$ by a straightforward induction on the derived length $d$. □

We now recall some definitions and notation from representation theory. If $V$ is a $kG$-module for a field $k$ and a group $G$, we use the right operator notation $vg$ for $v \in V$ and $g \in G$. We use the centralizer notation for fixed points, like $C_V(g) = \{v \in V \mid vg = v\}$. We also use the commutator notation $[v, g] = -v + vg$ for $v \in V$ and $g \in G$. The commutator subspaces are defined accordingly: if $B \leqslant G$, then $[V, B]$ is the span of all commutators $[v, b]$, where $v \in V$ and $b \in B$. The subspace $[V, B]$ coincides with the commutator subgroup $[V, B]$ in the natural semidirect product $VG$ when $V$ is regarded as the additive group acted upon by $G$. In particular, $[V, B]$ is $B$-invariant, and thus can be regarded as a $kB$-submodule.

For a group $G$ and a field $k$, a free $kG$-module of rank $n$ is a direct sum of $n$ copies of the group algebra $kG$ each of which is regarded as a vector space over $k$ of dimension $|G|$ with a basis $\{b_g \mid g \in G\}$ labelled by elements of $G$ on which $G$ acts in a regular permutation representation: $b_g h = b_{gh}$. In other words, a free $kG$-module $V = \bigoplus_{g\in G} V_g$ is a direct sum of subspaces that are regularly permuted by $G$ so that $V_g h = V_{gh}$.

The following lemma is known in the literature (see, for example, [25, Lemma 4.5]), but we give a proof for completeness.

**Lemma 2.4.** *Suppose that an abelian $p$-group $M$ is acted upon by a cyclic group $\langle\varphi\rangle$ of order $p^n$ and $V$ is a $kM\langle\varphi\rangle$-module for a field $k$ of characteristic different from $p$. If the subgroup $[M, \varphi^{p^{n-1}}]$ acts non-trivially on $V$, then the subspace $[V, [M, \varphi^{p^{n-1}}]]$ is a free $k\langle\varphi\rangle$-module.*

Here, of course, $\varphi^{p^{n-1}} = \varphi$ if $n = 1$.

*Proof.* The subspace $[V, [M, \varphi^{p^{n-1}}]]$ is clearly $M\langle\varphi\rangle$-invariant, so is an $kM\langle\varphi\rangle$-module. We extend the ground field to its algebraic closure $\bar{k}$ and denote by $W = V \otimes_k \bar{k}$ the resulting $\bar{k}M\langle\varphi\rangle$-module. Then $[W, [M, \varphi^{p^{n-1}}]]$ is a $\bar{k}M\langle\varphi\rangle$-module obtained from $[V, [M, \varphi^{p^{n-1}}]]$ by the field extension.

Since the characteristic of the ground field is coprime to $|M\langle\varphi\rangle|$, by Maschke's theorem

$$W = C_W([M, \varphi^{p^{n-1}}]) \oplus [W, [M, \varphi^{p^{n-1}}]]$$

is a completely reducible $\bar{k}M\langle\varphi\rangle$-module. Let $U$ be an irreducible $\bar{k}M\langle\varphi\rangle$-submodule of $[W, [M, \varphi^{p^{n-1}}]]$ on which $[M, \varphi^{p^{n-1}}]$ acts non-trivially.

By Clifford's theorem, $U = U_1 \oplus \cdots \oplus U_m$ decomposes into homogeneous $\bar{k}M$-submodules $U_i$ (Wedderburn components). The group $\langle\varphi\rangle$ transitively permutes the $U_i$. If the kernel of this permutational action was non-trivial, then $\varphi^{p^{n-1}}$ would stabilize all the $U_i$. But the abelian group $M$ acts by scalar transformations on each homogeneous component $U_i$. Hence $[M, \varphi^{p^{n-1}}]$ would act trivially on each $U_i$ and therefore on $U$, contrary to our assumption. Thus, $U$ is a free $\bar{k}\langle\varphi\rangle$-module.

Since $[W, [M, \varphi^{p^{n-1}}]]$ is the direct sum of such $U$, we obtain that $[W, [M, \varphi^{p^{n-1}}]]$ is also a free $\bar{k}\langle\varphi\rangle$-module. Then $[V, [M, \varphi^{p^{n-1}}]]$ is a free $k\langle\varphi\rangle$-module. Indeed, by the Deuring–Noether theorem [38, Theorem 29.7] two representations over a smaller field are equivalent if they are equivalent over a larger field. Being a free $\bar{k}\langle\varphi\rangle$-module, or a free $k\langle\varphi\rangle$-module, means having a basis, as of a vector space over the corresponding field, elements of which are permuted by $\varphi$ so that all orbits are regular. In such a basis $\langle\varphi\rangle$ is represented by the corresponding permutational matrices, all of which are defined over $k$. □

## 3. Automorphism of order $p^n$

First we state separately the following proposition, which will also be used in the next section in a different situation.

**Proposition 3.1.** *Suppose that a cyclic group $\langle\varphi\rangle$ of order $p^n$ acts by automorphisms on a finite $p$-group $P$, and $V$ is a faithful $\mathbb{F}_q P\langle\varphi\rangle$-module, where $\mathbb{F}_q$ is a prime field of order $q \neq p$. Then the derived length of $[P, \varphi^{p^{n-1}}]$ is bounded in terms of $|C_V(\varphi)|$ and $p^n$.*

*Proof.* Let $M$ be a maximal abelian normal subgroup of the semidirect product $P\langle\varphi\rangle$. If $[M, \varphi^{p^{n-1}}] \neq 1$, then by Lemma 2.4, $[V, [M, \varphi^{p^{n-1}}]]$ is a free $\mathbb{F}_q\langle\varphi\rangle$-module. Obviously, in a free $\mathbb{F}_q\langle\varphi\rangle$-module the fixed points of $\varphi$ are exactly the "diagonal" elements. Hence the order of $[V, [M, \varphi^{p^{n-1}}]]$ is equal to

$$|C_{[V,[M,\varphi^{p^{n-1}}]]}(\varphi)|^{|\varphi|} = |C_{[V,[M,\varphi^{p^{n-1}}]]}(\varphi)|^{p^n}$$

and therefore is bounded in terms of $|C_V(\varphi)|$ and $p^n$. The group $[M, \varphi^{p^{n-1}}]$ acts faithfully on $V$; therefore by Maschke's theorem it also acts faithfully on $[V, [M, \varphi^{p^{n-1}}]]$. Hence the order of $[M, \varphi^{p^{n-1}}]$ is bounded in terms of $|C_V(\varphi)|$ and $p^n$. The same of course holds if $[M, \varphi^{p^{n-1}}] = 1$.

It follows that the index $|M : C_M(\varphi^{p^{n-1}})|$ is bounded in terms of $|C_V(\varphi)|$ and $p^n$, since this index is equal to the number of different commutators $[m, \varphi^{p^{n-1}}]$ for $m \in M$.

Consider a central series of $P\langle\varphi\rangle$ connecting 1 and $M$. Since $|M : C_M(\varphi^{p^{n-1}})|$ is bounded in terms of $|C_V(\varphi)|$ and $p^n$, the number of factors of this series that are not covered by $C_M(\varphi^{p^{n-1}})$ is bounded in terms of $|C_V(\varphi)|$ and $p^n$. Therefore there is a normal series of bounded length connecting 1 and $M$ each factor of which is either central in $P\langle\varphi\rangle$ or is covered by $C_M(\varphi^{p^{n-1}})$. Obviously, then $\varphi^{p^{n-1}}$ acts trivially on each factor of this series, and therefore so does $[P, \varphi^{p^{n-1}}]$. By Kaluzhnin's theorem, the automorphism group induced by the action of $[P, \varphi^{p^{a-1}}]$ on $M$ is nilpotent of bounded class. Since $M$ contains its centralizer in $P\langle\varphi\rangle$, it follows that $[P, \varphi^{p^{n-1}}]$ is soluble of bounded derived length, since by the above $\gamma_s([P, \varphi^{p^{n-1}}]) \leqslant M \cap [P, \varphi^{p^{n-1}}]$ for some number $s$ bounded in terms of $|C_V(\varphi)|$ and $p^n$. □

*Proof of Theorem* 1.1. Recall that $G$ is a finite $p$-soluble group admitting an automorphism $\varphi$ of order $p^n$ such that $\varphi$ has at most $m$ fixed points in every $\varphi$-invariant elementary abelian $p'$-section of $G$. We need to bound the $p$-length of $G$ in terms of $p^n$ and $m$. Henceforth in this section, saying for brevity that a certain parameter is simply "bounded" we mean that this parameter is bounded above in terms of $p^n$ and $m$.

We use induction on $n$. It is convenient to consider the case of $n = 0$ as the basis of induction, when $|\varphi| = p^0 = 1$, that is, $\varphi$ acts trivially on $G$. Then the hypothesis means that every elementary abelian $p'$-section of $G$ has bounded order. We claim that the nilpotency class of a Sylow $p$-subgroup $P$ of $\hat{G} = G/O_p(G)$ is bounded. Indeed, since the order of $P$ is coprime to $|O_{p'}(\hat{G})|$, for every prime $q$ dividing $|O_{p'}(\hat{G})|$ there is a $P$-invariant Sylow $q$-subgroup $Q$ of $O_{p'}(\hat{G})$. The quotient $P/C_P(Q)$ acts faithfully on the Frattini quotient $Q/\Phi(Q)$, which has order at most $m$ by the assumption. Hence $P/C_P(Q)$ has bounded order and therefore bounded nilpotency class. Since $P$ acts faithfully on $O_{p'}(\hat{G})$, we have $\bigcap C_P(Q_i) = 1$, where $Q_i$ runs over all $P$-invariant Sylow subgroups of $O_{p'}(\hat{G})$. Hence $P$ is a subdirect product of groups of bounded nilpotency class and therefore has bounded nilpotency class itself. We now obtain that the $p$-length of $\hat{G} = G/O_p(G)$ is bounded by the Hall–Higman theorem [32]. As a result, the $p$-length of $G$ is bounded.

From now on we assume that $n \geqslant 1$.

Let $\hat{G} = G/O_p(G)$. Consider a Sylow $p$-subgroup of the semidirect product $\hat{G}\langle\varphi\rangle$ containing $\langle\varphi\rangle$ and let $P$ be its intersection with $\hat{G}$, so that $P$ is a $\varphi$-invariant Sylow $p$-subgroup of $\hat{G}$. Since the order of the $p$-group $P\langle\varphi\rangle$ is coprime to $|O_{p'}(\hat{G})|$, for every prime $q$ dividing $|O_{p'}(\hat{G})|$ there is a $P\langle\varphi\rangle$-invariant Sylow $q$-subgroup $Q$ of $O_{p'}(\hat{G})$.

The quotient $\bar{P} = P/C_P(Q)$ acts faithfully on the Frattini quotient $V = Q/\Phi(Q)$, which we regard as an $\mathbb{F}_q P\langle\varphi\rangle$-module. By hypothesis, $|C_V(\varphi)| \leqslant m$, so by Proposition 3.1 the derived length of $[\bar{P}, \varphi^{p^{n-1}}]$ is bounded. In other words, $[P, \varphi^{p^{n-1}}]^{(s)} \leqslant C_P(Q)$ for some bounded number $s$. Since $P$ acts faithfully on $O_{p'}(\hat{G})$, we have $\bigcap C_P(Q_i) = 1$, where $Q_i$ runs over all $P\langle\varphi\rangle$-invariant Sylow subgroups of $O_{p'}(\hat{G})$. Hence, $[P, \varphi^{p^{n-1}}]^{(s)} = 1$.

By the Hall–Higman–Hartley Theorem 2.3 we now obtain that the normal subgroup $[P, \varphi^{p^{n-1}}]$ of the Sylow $p$-subgroup $P$ is contained in $H = O_{p',p,p',\dots,p',p}(\hat{G})$, where $p$ occurs boundedly many times.

Consider the action of $\varphi$ on the quotient $\tilde{G} = \hat{G}/H$. Since $[P, \varphi^{p^{n-1}}] \leqslant H$, it follows that $\varphi^{p^{n-1}}$ acts trivially on the image of $P$, which is a Sylow $p$-subgroup of $\tilde{G}$. In particular, $\varphi^{p^{n-1}}$ acts trivially on $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$, and therefore so does $[\tilde{G}, \varphi^{p^{n-1}}]$. Since $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$ contains its centralizer in $\tilde{G}/O_{p'}(\tilde{G})$, we obtain that $[\tilde{G}, \varphi^{p^{n-1}}] \leqslant O_{p',p}(\tilde{G})$. In other words, $\varphi^{p^{n-1}}$ acts trivially on the quotient $\tilde{G}/O_{p',p}(\tilde{G})$. Therefore the order of the automorphism induced by $\varphi$ on $\tilde{G}/O_{p',p}(\tilde{G})$ is at most $p^{n-1}$. By the induction hypothesis the $p$-length of this quotient is bounded. Then the $p$-length of $G/O_{p,p'}(G)$ is bounded, and therefore the $p$-length of $G$ is bounded, as required. □

*Proof of Corollary* 1.3. Here, $G$ is a finite soluble group admitting an automorphism $\varphi$ of order $p^n$ such that $\varphi$ has at most $m$ fixed points in every $\varphi$-invariant elementary abelian $p'$-section of $G$. By Theorem 1.1 the $p$-length of $G$ is bounded. It remains to obtain a bound for the Fitting height of every $p'$-factor $T$ of the upper $p$-series consisting of the subgroups $O_{p',p,p',p,\dots}$. It is known that the rank of a finite group is bounded in terms of the ranks of its elementary abelian sections. Here, by definition, the rank of a group is the minimum number $r$ such that every subgroup can be generated by $r$ elements. Of course every elementary abelian section of $C_T(\varphi)$ is a $\varphi$-invariant $p'$-section of $G$ and therefore has bounded order by hypothesis. It is also known that the Fitting height of a soluble finite group is bounded in terms of its rank. Thus $C_T(\varphi)$ has bounded Fitting height and therefore so does $G$ by Thompson's theorem [16]. □

**Remark 3.2.** If we could obtain in Theorem 1.1 a "strong" bound for the $p$-length, in terms of $\alpha(\langle\varphi\rangle)$ only, for a subgroup of bounded index, then a similar strong bound could be obtained in Corollary 1.3 for the Fitting height of a subgroup of bounded index. This would follow from a rank analogue of the Hartley–Isaacs theorem proved in [39], which states that if a finite soluble group $K$ admits a soluble group of automorphisms $L$ of coprime order, then $K$ has a normal subgroup $N$ of Fitting height at most $5(4^{\alpha(L)}-1)/3$ such that the order of $K/N$ is bounded in terms of $|L|$ and the rank of $C_K(L)$.

## 4. Automorphism of order $p^a q^b$

*Proof of Theorem* 1.4. Recall that $G$ is a finite group admitting an automorphism $\varphi$ of order $p^a q^b$. By Hartley's theorem [3] (based on the classification of finite simple groups), $G$ has a soluble subgroup of index bounded in terms of $p^a q^b$ and $|C_G(\varphi)|$. Therefore we can assume from the outset that $G$ is soluble, so that we need to bound the Fitting height of $G$ in terms of $p^a q^b$ and $|C_G(\varphi)|$. Throughout this section we say for brevity that a certain parameter is "bounded" meaning that this parameter is bounded above in terms of $p^a q^b$ and $|C_G(\varphi)|$. We use without special references the fact that the number of fixed points of $\varphi$ in every $\varphi$-invariant section of $G$ is at most $|C_G(\varphi)|$ by Lemma 2.1.

We use induction on $a+b$. As a basis of induction we consider the case when either $a=0$ or $b=0$. Then $|\varphi|$ is a prime-power, and by the Hartley–Turau theorem [25] the group $G$ has a subgroup of bounded index that has Fitting height at most $\alpha(\varphi)$. (Actually, for our 'weak' bound a simpler argument would suffice: if, say, $|\varphi| = p^a$, then the rank of the Frattini quotient of $O_{p',p}(G)/O_{p'}(G)$ is bounded by Lemma 2.2, which implies a bound for the Fitting height of $G/O_{p'}(G)$, and the Fitting height of $O_{p'}(G)$ is bounded in terms of $a$ by Thompson's theorem [16].) Moreover, the following proposition holds, which apparently was noted by Hartley but may have remained unpublished. We state this proposition in a more general form, without assuming that the automorphism has biprimary order.

**Proposition 4.1.** *If a finite soluble group $G$ admits an automorphism $\psi$ such that there is at most one prime dividing both $|\psi|$ and $|G|$, then the Fitting height of $G$ is bounded above in terms of $|\psi|$ and $|C_G(\psi)|$.*

*Proof.* If $(|\psi|,|G|)=1$, then the result follows from the stronger theorem of Thompson [16]. Now let $\langle\psi\rangle = \langle\psi_r\rangle \times \langle\psi_{r'}\rangle$, where $\langle\psi_r\rangle$ is the Sylow $r$-subgroup of $\langle\psi\rangle$ and $r$ is the only common prime divisor of $|G|$ and $|\psi|$. The centralizer $C_G(\psi_{r'})$ admits the automorphism $\psi_r$ of prime-power order whose centralizer $C_{C_G(\psi_{r'})}(\psi_r)$ is equal to $C_G(\psi)$. By the Hartley–Turau theorem, the Fitting height of $C_G(\psi_{r'})$ is bounded. We now apply Thompson's theorem to the automorphism $\psi_{r'}$ of $G$ of coprime order to obtain that the Fitting height of $G$ is bounded as required. □

We return to the proof of Theorem 1.4. Let $a \geqslant 1$ and $b \geqslant 1$. Let $\varphi_p = \varphi^{q^b}$ and $\varphi_q = \varphi^{p^a}$, so that $|\varphi_p| = p^a$ and $|\varphi_q| = q^b$, while $\langle\varphi\rangle = \langle\varphi_p\rangle \times \langle\varphi_q\rangle$. The subgroup $O_{q'}(G)$ admits the automorphism $\varphi$ whose order has at most one prime divisor $p$ in common with $|O_{q'}(G)|$. By Proposition 4.1 the Fitting height of $O_{q'}(G)$ is bounded.

Therefore we can assume that $O_{q'}(G)=1$. Then the quotient $\bar G = G/O_q(G)$ acts faithfully on the Frattini quotient $V = O_q(G)/\Phi(O_q(G))$, which we regard as an $\mathbb{F}_q\bar G\langle\varphi\rangle$-module. The fixed-point subspace $C_V(\varphi_p)$ has bounded order. This follows from Lemma 2.2 applied to the action of the linear transformation $\varphi_q$ of order $q^b$ on $C_V(\varphi_p)$, since the fixed points of $\varphi_q$ in $C_V(\varphi_p)$ are contained in the fixed-point subspace $C_V(\varphi)$ of bounded order.

Choose a Sylow $p$-subgroup of $\bar G\langle\varphi_p\rangle$ containing $\langle\varphi_p\rangle$, and let $P$ be its intersection with $\bar G$, so that $P$ is a $\varphi_p$-invariant Sylow $p$-subgroup of $\bar G$. By Proposition 3.1, the subgroup $[P, \varphi_p^{p^{a-1}}]$ has bounded derived length.

Hence by the Hall–Higman–Hartley Theorem 2.3 the normal subgroup $[P, \varphi_p^{p^{a-1}}]$ of the Sylow $p$-subgroup $P$ is contained in $H = O_{p',p,p',\dots,p',p}(\bar{G})$, where $p$ occurs boundedly many times.

Consider the action of $\varphi$ on the quotient $\tilde{G} = \bar{G}/H$. Since $[P, \varphi_p^{p^{a-1}}] \leqslant H$, it follows that $\varphi_p^{p^{a-1}}$ acts trivially on the image of $P$, which is a Sylow $p$-subgroup of $\tilde{G}$. In particular, $\varphi_p^{p^{a-1}}$ acts trivially on $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$, and therefore so does $[\tilde{G}, \varphi_p^{p^{a-1}}]$. Since $O_{p',p}(\tilde{G})/O_{p'}(\tilde{G})$ contains its centralizer in $\tilde{G}/O_{p'}(\tilde{G})$, we obtain that $[\tilde{G}, \varphi_p^{p^{a-1}}] \leqslant O_{p',p}(\tilde{G})$. In other words, $\varphi_p^{p^{a-1}}$ acts trivially on the quotient $\tilde{G}/O_{p',p}(\tilde{G})$. Therefore the order of the automorphism induced by $\varphi$ on $\tilde{G}/O_{p',p}(\tilde{G})$ divides $p^{a-1}q^b$. By induction, the Fitting height of this quotient is bounded.

It remains to obtain a bound for the Fitting height of each of the boundedly many $\varphi$-invariant normal $p'$-sections that appear in the upper $p$-series of the groups $H$ and $O_{p',p}(\tilde{G})$. Such a bound follows from Proposition 4.1. □

*Proof of Corollary* 1.5. This corollary for locally finite groups follows from Theorem 1.4 by the standard inverse limit argument. □

*Proof of Corollary* 1.6. Here, a finite group $G$ admits an automorphism $\varphi$ such that there are at most two primes dividing both $|\varphi|$ and $|G|$. Again, by Hartley's theorem [3] we can assume from the outset that $G$ is soluble. If $(|\varphi|, |G|)$ is 1 or a prime power, then the result follows from Proposition 4.1. Now let $\langle\varphi\rangle = \langle\varphi_{pq}\rangle \times \langle\psi\rangle$, where $\langle\varphi_{pq}\rangle$ is the Hall $\{p, q\}$-subgroup of $\langle\varphi\rangle$ and $p, q$ are the only common prime divisors of $|G|$ and $|\varphi|$. The centralizer $C_G(\psi)$ admits the automorphism $\varphi_{pq}$ of biprimary order whose centralizer $C_{C_G(\psi)}(\varphi_{pq})$ is equal to $C_G(\varphi)$. By Theorem 1.4, the Fitting height of $C_G(\psi)$ is bounded in terms of $|\varphi_{pq}|$ and $|C_G(\varphi)|$. We now apply Thompson's theorem [16] to the automorphism $\psi$ of $G$ of coprime order to obtain that the Fitting height of $G$ is bounded in terms of $|\varphi|$ and $|C_G(\varphi)|$. □

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