

Frobenius groups of automorphisms and their fixed points

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Abstract. Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that the fixed-point subgroup of F is trivial: $C_G(F) = 1$. In this situation various properties of G are shown to be close to the corresponding properties of $C_G(H)$. By using Clifford's theorem it is proved that the order $|G|$ is bounded in terms of $|H|$ and $|C_G(H)|$, the rank of G is bounded in terms of $|H|$ and the rank of $C_G(H)$, and that G is nilpotent if $C_G(H)$ is nilpotent. Lie ring methods are used for bounding the exponent and the nilpotency class of G in the case of metacyclic FH . The exponent of G is bounded in terms of $|FH|$ and the exponent of $C_G(H)$ by using Lazard's Lie algebra associated with the Jennings–Zassenhaus filtration and its connection with powerful subgroups. The nilpotency class of G is bounded in terms of $|H|$ and the nilpotency class of $C_G(H)$ by considering Lie rings with a finite cyclic grading satisfying a certain 'selective nilpotency' condition. The latter technique also yields similar results bounding the nilpotency class of Lie rings and algebras with a metacyclic Frobenius group of automorphisms, with corollaries for connected Lie groups and torsion-free locally nilpotent groups with such groups of automorphisms. Examples show that such nilpotency results are no longer true for non-metacyclic Frobenius groups of automorphisms.

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1 Introduction

Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that the fixed-point subgroup (which we call the centralizer) of F is trivial: $C_G(F) = 1$. Experience shows that many prop-

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erties of G must be close to the corresponding properties of $C_G(H)$. For example, when GF is also a Frobenius group with kernel G and complement F (so that GFH is a double Frobenius group), the second and third authors [19] proved that the nilpotency class of G is bounded in terms of $|H|$ and the nilpotency class of $C_G(H)$. This result solved in the affirmative Mazurov's Problem 17.72 (a) in Kourovka Notebook [25].

In this paper we derive properties of G from the corresponding properties of $C_G(H)$ in more general settings, no longer assuming that GF is also a Frobenius group. Note also that the order of G is not assumed to be coprime to the order of FH .

By using variations on Clifford's theorem it is shown that the order $|G|$ is bounded in terms of $|H|$ and $|C_G(H)|$, the rank of G is bounded in terms of $|H|$ and the rank of $C_G(H)$, and that G is nilpotent if $C_G(H)$ is nilpotent. By using various Lie ring methods bounds for the exponent and the nilpotency class of G are obtained in the case of metacyclic FH . For bounding the exponent, we use Lazard's Lie algebra associated with the Jennings–Zassenhaus filtration and its connection with powerful subgroups. For bounding the nilpotency class we consider Lie rings with a finite cyclic grading satisfying a certain condition of 'selective nilpotency'. The latter technique yields similar results giving nilpotency of bounded nilpotency class (also known as nilpotency index) of Lie rings and algebras with a metacyclic Frobenius group of automorphisms, with corollaries for connected Lie groups and torsion-free locally nilpotent groups with such groups of automorphisms. Examples show that such nilpotency results are no longer true for non-metacyclic Frobenius groups of automorphisms.

We now describe the results and the structure of the paper in more detail. Recall that we consider a finite group G admitting a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. It is worth noting from the outset that since F is nilpotent being a Frobenius kernel, the condition $C_G(F) = 1$ implies the solvability of G by a theorem of Belyaev and Hartley [1] based on the classification of finite simple groups.

In Section 2 we begin establishing the connection between the properties of G and $C_G(H)$ by proving that the orders satisfy the equation $|G| = |C_G(H)|^{|H|}$, the rank of G is bounded in terms of $|H|$ and the rank of $C_G(H)$, and that G is nilpotent if $C_G(H)$ is nilpotent (Theorem 2.7). These results are proved by using Clifford's theorem on the basis of information about the fixed points of H in FH -invariant sections of G . In particular, we prove that these fixed points are the images of elements of $C_G(H)$ (Theorem 2.3), which is a non-trivial fact since the orders of G and H are not assumed to be coprime.

In Section 3 we deal with bounding the exponent. First we develop the requisite Lie ring technique, some of which was used earlier by the first and third authors

in [11]. We define Lazard's Lie algebra $L_p(P)$ associated with the Jennings–Zassenhaus filtration of a finite p -group P and recall Lazard's observation that elements of P of order p^k give rise to elements of $L_p(P)$ that are ad-nilpotent of index p^k . A useful property is the existence of a powerful subgroup of P of index bounded in terms of the number of generators of P and the nilpotency class of $L_p(P)$. A key lemma gives a bound for the nilpotency class of a finitely generated solvable Lie algebra 'saturated' with ad-nilpotent elements. The main result of the section is Theorem 3.4: if a finite group G admits a Frobenius group of automorphisms FH with cyclic kernel F and complement H such that $C_G(F) = 1$, then the exponent of G is bounded in terms of $|FH|$ and the exponent of $C_G(H)$. At present it is unclear, even in the case where GFH is a double Frobenius group, if the bound can be made independent of $|F|$, which would give an affirmative answer to part (b) of Mazurov's Problem 17.72 in [25]. In the proof of Theorem 3.4 a reduction to finite p -groups is given by Dade's theorem [3]. Then nilpotency of Lazard's Lie algebra gives a reduction to powerful p -groups, to which a lemma from Section 2 about fixed points of H is applied.

In Section 4 a different Lie ring theory is developed, which is used later in Section 5 for bounding the nilpotency class of groups and Lie rings with a metacyclic Frobenius group FH of automorphisms. This theory is stated in terms of a Lie ring L with a finite cyclic grading (which naturally arises from the 'eigenspaces' for F). The condition of the fixed-point subring $C_L(H)$ being nilpotent of class c implies certain restrictions on the commutation of the grading components, which we nickname 'selective nilpotency'. For example, in [9] it was shown that if we have $C_L(F) = 0$, $c = 1$, and $|F|$ is a prime, then each component commutes with all but at most $(c, |H|)$ -boundedly many components, which in turn implies a strong bound for the nilpotency class of L . For greater values of c more complicated 'selective nilpotency' conditions naturally arise; similar conditions were exploited earlier in the paper [19] on double Frobenius groups.

In Section 5 we obtain bounds for the nilpotency class of groups and Lie rings with metacyclic Frobenius groups of automorphisms. (Examples show that such results are no longer true for non-metacyclic Frobenius groups of automorphism.) The main result for finite groups is Theorem 5.8: if a finite group G admits a Frobenius group of automorphisms FH with cyclic kernel F and complement H such that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class c , then G is nilpotent of class bounded in terms of c and $|H|$ only. The proof is based on the analogous result for Lie rings (Theorem 5.6).

We state separately the result for Lie algebras as Theorem 5.1: if a Lie algebra L over any field admits a Frobenius group of automorphisms FH with cyclic kernel F such that $C_L(F) = 0$ and $C_L(H)$ is nilpotent of class c , then L is nilpotent of class bounded in terms of c and $|H|$ only. (Here $C_L(F)$ and $C_L(H)$ denote the

fixed-point subalgebras for F and H .) One corollary of this theorem is for connected Lie groups with metacyclic Frobenius groups of automorphisms satisfying similar conditions (Theorem 5.4). Another application is for torsion-free locally nilpotent groups with such groups of automorphisms (Theorem 5.5).

The induced group of automorphisms of an invariant section is often denoted by the same letter (which is a slight abuse of notation as the action may become non-faithful). We use the abbreviation, say, “ (m, n) -bounded” for “bounded above in terms of m, n only”.

2 Fixed points of Frobenius complements

We begin with a theorem of Belyaev and Hartley [1] based on the classification of finite simple groups (see also [14]).

Theorem 2.1 ([1, Theorem 0.11]). *Suppose that a finite group G admits a nilpotent group of automorphisms F such that $C_G(F) = 1$. Then G is solvable.*

We now discuss the question of covering the fixed points of a group of automorphisms in an invariant quotient by the fixed points in the group. Let $A \leq \text{Aut } G$ for a finite group G and let N be a normal A -invariant subgroup of G . It is well known that if $(|A|, |N|) = 1$, then $C_{G/N}(A) = C_G(A)N/N$. If we do not assume that $(|A|, |N|) = 1$, the equality $C_{G/N}(A) = C_G(A)N/N$ may no longer be true. However there are some important cases when it does hold. In particular, we have the following lemma.

Lemma 2.2. *Let G be a finite group admitting a nilpotent group of automorphisms F such that $C_G(F) = 1$. If N is a normal F -invariant subgroup of G , then*

$$C_{G/N}(F) = 1.$$

Proof. Since F is a Carter subgroup of GF , it follows that NF/N is a Carter subgroup of GF/N . Hence $C_{G/N}(F) = 1$. \square

The following theorem was proved in [10] under the additional coprimeness assumption $(|N|, |F|) = 1$, so here we only have to provide a reduction to this case.

Theorem 2.3. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H . If N is an FH -invariant normal subgroup of G such that $C_N(F) = 1$, then*

$$C_{G/N}(H) = C_G(H)N/N.$$

Proof. As a Frobenius kernel, F is nilpotent. Hence N is solvable by Theorem 2.1. Consider an unrefinable FH -invariant normal series of G

$$G > N = N_1 > N_2 > \dots > N_k > N_{k+1} = 1$$

connecting N with 1; its factors N_i/N_{i+1} are elementary abelian. We apply induction on k to find an element of $C_G(H)$ in any $gN \in C_{G/N}(H)$.

For $k > 1$ consider the quotient G/N_k and the induced group of automorphisms FH . By Lemma 2.2,

$$C_{G/N_k}(F) = 1.$$

By induction there is $c_1N_k \in C_{G/N_k}(H) \cap gN/N_k$, and it remains to find a required element $c \in C_G(H) \cap c_1N_k \subseteq C_G(H) \cap gN$. Thus the proof of the induction step will follow from the case $k = 1$.

Let $k = 1$; then $N = N_k$ is a p -group for some prime p . Let $F = F_p \times F_{p'}$, where F_p is the Sylow p -subgroup of F . Since $C_N(F_{p'})$ is F_p -invariant, we must actually have $C_N(F_{p'}) = 1$. Indeed, otherwise the p -group F_p would have non-trivial fixed points on the p -group $C_N(F_{p'})$, and clearly $C_{C_N(F_{p'})}(F_p) = C_N(F)$ (this argument works even if $F_p = 1$). Thus, the hypotheses of the theorem also hold for G with the Frobenius group of automorphisms $F_{p'}H$ satisfying the additional condition $(|N|, |F_{p'}|) = 1$. Now [10, Theorem 1] can be applied to produce a required fixed point. □

We now prove a few useful lemmas about a finite group with a Frobenius group of automorphisms.

Lemma 2.4. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then we have $G = \langle C_G(H)^f \mid f \in F \rangle$.*

Proof. The group G is solvable by Theorem 2.1. Consider an unrefinable FH -invariant normal series

$$G = G_1 > G_2 > \dots > G_k > G_{k+1} = 1. \tag{2.1}$$

It is clearly sufficient to prove that every factor $S = G_i/G_{i+1}$ of this series is covered by $\langle C_{G_i}(H)^f \mid f \in F \rangle$, that is,

$$\langle C_{G_i}(H)^f \mid f \in F \rangle G_{i+1}/G_{i+1} = G_i/G_{i+1}.$$

By Theorem 2.3, this is the same as $\langle C_S(H)^f \mid f \in F \rangle = S$. Recall that we have $C_S(F) = 1$ by Lemma 2.2. Then Clifford's theorem can be applied to show that $C_S(H) \neq 1$.

Recall that for a group A and a field k , a free kA -module of dimension n is a direct sum of n copies of the group algebra kA , each of which can be regarded as a vector space over k of dimension $|A|$ with a basis $\{v_g \mid g \in A\}$ labelled by elements of A on which A acts in a regular representation: $v_g h = v_{gh}$. By the Deuring–Noether theorem [2, Theorem 29.7] two representations over a smaller field are equivalent if they are equivalent over a larger field; therefore being a free kA -module is equivalent to being a free $\bar{k}A$ -module for any field extension $\bar{k} \supseteq k$, as the corresponding permutational matrices are defined over the prime field.

We denote by \mathbb{F}_p the field of p elements.

Lemma 2.5. *Each factor S of (2.1) is a free $\mathbb{F}_p H$ -module for the appropriate prime p .*

Proof. Again, we only provide reduction to the coprime case considered in [10]. Let S be an elementary p -group; then let $F = F_p \times F_{p'}$ as in the proof of Theorem 2.3 above. As therein, we must actually have $C_S(F_{p'}) = 1$. Refining S by a non-refinable $F_{p'}H$ -invariant normal series we obtain factors that are irreducible $\mathbb{F}_p F_{p'}H$ -modules. Having the additional condition that $p \nmid |F_{p'}|$ we can now apply, for example, [10, Lemma 2] to obtain that each of them is a free $\mathbb{F}_p H$ -module, and therefore S is also a free $\mathbb{F}_p H$ -module. \square

We now finish the proof of Lemma 2.4. By Lemma 2.5, S is a free $\mathbb{F}_p H$ -module, which means that $S = \bigoplus_{h \in H} Th$ for some $\mathbb{F}_p H$ -submodule T . Hence we have $C_S(H) \neq 0$, as $0 \neq \sum_{h \in H} th \in C_S(H)$ for any $0 \neq t \in T$. Since the series (2.1) is non-refinable, the $\mathbb{F}_p FH$ -module S is irreducible. Therefore,

$$0 \neq \langle C_S(H)^{HF} \rangle = \langle C_S(H)^F \rangle = S,$$

which is exactly what we need. \square

Lemma 2.6. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then for each prime p dividing $|G|$ there is a unique FH -invariant Sylow p -subgroup of G .*

Proof. Recall that G is solvable, and so is GF . As F is a Carter subgroup of GF , it contains a system normalizer of G . By P. Hall's theorem [22, Theorem 9.2.6], a system normalizer covers all central factors of any chief series of GF . Since F is nilpotent, it follows that F is a system normalizer. Furthermore, F normalizes a unique Sylow p -subgroup. Indeed, if P and P^g for $g \in G$ are two Sylow p -subgroups normalized by F , then P is normalized by F and $F^{g^{-1}}$. Then F and $F^{g^{-1}}$ are Carter subgroups of $N_G(P)$ and $F = F^{g^{-1}n}$ for some $n \in N_G(P)$, whence $g^{-1}n = 1$, as $N_G(F) = C_G(F) = 1$. Thus, $P^g = P^n = P$. Since F is normal in FH , the uniqueness of P implies that it is also H -invariant. \square

We now establish the connection between the order, rank, and nilpotency of G and $C_G(H)$ for a finite group G admitting a Frobenius group of automorphisms FH with fixed-point-free kernel F . (By the rank we mean the minimum number r such that every subgroup can be generated by r elements.)

Theorem 2.7. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then*

- (a) $|G| = |C_G(H)|^{|H|}$,
- (b) *the rank of G is bounded in terms of $|H|$ and the rank of $C_G(H)$,*
- (c) *if $C_G(H)$ is nilpotent, then G is nilpotent.*

Proof. (a) It is sufficient to prove this equality for each factor $S = G_i/G_{i+1}$ of the series (2.1), since

$$|C_G(H)| = \prod_i |C_{G_i/G_{i+1}}(H)|$$

by Theorem 2.3. By Lemma 2.5, S is a free $\mathbb{F}_p H$ -module, which means that

$$S = \bigoplus_{h \in H} Th$$

for some $\mathbb{F}_p H$ -submodule T . Therefore, we have $C_S(H) = \{\sum_{h \in H} th \mid t \in T\}$ and $|C_S(H)| = |T|$, whence $|S| = |T|^{|H|}$.

(b) It is known that the rank of a finite (solvable) group G is bounded in terms of the maximum rank of its Sylow subgroups ([12]). Let P be an FH -invariant Sylow p -subgroup of G given by Lemma 2.6. It is known that the rank of a p -group of automorphisms of a finite p -group U is bounded in terms of the rank of U . Let U denote a Thompson critical subgroup of P ; recall that U is a characteristic subgroup of nilpotency class at most 2 containing its centralizer in P (see, for example, [5, Theorem 5.3.11]). Thus the rank of P is bounded in terms of the rank of U . In turn, since U is nilpotent of class at most 2, the rank of U is bounded in terms of the rank of $S = U/\Phi(U)$. The group S can be regarded as an $\mathbb{F}_p FH$ -module, which is a free $\mathbb{F}_p H$ -module by a repeated application of Lemma 2.5 to an unrefinable series of $\mathbb{F}_p FH$ -submodules of $U/\Phi(U)$. By the same argument as in the proof of (a) above, the rank of S is equal to $|H|$ times the rank of $C_S(H)$. By Theorem 2.3, $C_S(H)$ is covered by $C_G(H)$; as a result, the rank of S is at most $|H|$ times the rank of $C_G(H)$.

(c) First we make a simple remark that in any action of the Frobenius group FH with non-trivial action of F the complement H acts faithfully. Indeed, the kernel K that does not contain F must intersect H trivially: $K \cap H$ acts trivially on

$F/(K \cap F) \neq 1$ and therefore has non-trivial fixed points on F , as the action is coprime.

Suppose that $C_G(H)$ is nilpotent. We prove that then G is nilpotent by contradiction, considering a counterexample GFH of minimal possible order. Recall that G is solvable. Suppose that G is not nilpotent; then it is easy to find an FH -invariant section VU of G such that V and U are elementary abelian groups of coprime orders, V is normal in $VUFH$, and U acts faithfully on V with $C_V(U) = 1$. Note that $C_{VU}(F) = 1$ by Lemma 2.2. In particular, F acts non-trivially on VU and therefore H acts faithfully on VU . Since $C_{VU}(H)$ is covered by $C_G(H)$ by Theorem 2.3, it follows that $C_{VU}(H)$ is nilpotent. Thus, we can replace G by UV , and F by its image in its action on UV , so by the minimality of our counterexample we must actually have $G = VU$.

Also by Theorem 2.3, $C_{VU}(H) = C_V(H)C_U(H)$. Furthermore, since U and V have coprime orders, the nilpotency of $C_{VU}(H)$ implies that it is abelian; in other words, $C_U(H)$ centralizes $C_V(H)$. Note that $C_U(H) \neq 1$ by Lemma 2.4 (or 2.5).

Let V be an elementary p -group; we can regard V as an $\mathbb{F}_p UFH$ -module. Note that V is a free $\mathbb{F}_p H$ -module by Lemma 2.5. We extend the ground field \mathbb{F}_p to a finite field $\overline{\mathbb{F}}_p$ that is a splitting field for UFH and obtain an $\overline{\mathbb{F}}_p UFH$ -module

$$\widetilde{V} = V \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p.$$

Many of the above-mentioned properties of V are inherited by \widetilde{V} :

- (V1) \widetilde{V} is a faithful $\overline{\mathbb{F}}_p U$ -module,
- (V2) $C_{\widetilde{V}}(U) = 0$,
- (V3) $C_U(H)$ acts trivially on $C_{\widetilde{V}}(H)$,
- (V4) $C_{\widetilde{V}}(F) = 0$,
- (V5) \widetilde{V} is a free $\overline{\mathbb{F}}_p H$ -module.

Consider an unrefinable series of $\overline{\mathbb{F}}_p UFH$ -submodules

$$\widetilde{V} = V_1 > V_2 > \cdots > V_k > V_{k+1} = 0. \quad (2.2)$$

Suppose that W is one of the factors of this series; it is a non-trivial irreducible $\overline{\mathbb{F}}_p UFH$ -submodule. Note that $C_W(F) = 0$ by Lemma 2.2, as we can still regard \widetilde{V} as a finite (additive) group on which F acts fixed-point-freely by property (V4). For the same reason, W is a free $\overline{\mathbb{F}}_p H$ -module by Lemma 2.5.

Let

$$W = W_1 \oplus \cdots \oplus W_t$$

be the decomposition of W into the direct sum of Wedderburn components W_i with respect to U . On each of the W_i the group U is represented by scalar multiplications. We consider the transitive action of FH on the set $\Omega = \{W_1, \dots, W_t\}$.

Lemma 2.8. *All orbits of H on Ω , except for possibly one, are regular (that is, of length $|H|$).*

Proof. First note that H transitively permutes the F -orbits on Ω . Let Ω_1 be one of these F -orbits and let H_1 be the stabilizer of Ω_1 in H . If $H_1 = 1$, then all the H -orbits are regular, so we assume that $H_1 \neq 1$. We claim that H_1 has exactly one non-regular orbit on Ω_1 (actually, a fixed point).

Let \bar{F} be the image of F in its action on Ω_1 . If $\bar{F} = 1$, then Ω_1 consists of a single Wedderburn component, on which U acts by scalar multiplications, and acts non-trivially by property (V2). Then F acts trivially on the non-trivial quotient of U by the corresponding kernel, which contradicts Lemma 2.2. Thus, $\bar{F} \neq 1$, and by the remark at the beginning of the proof, $\bar{F}H_1$ is a Frobenius group with complement H_1 .

Let S be the stabilizer of a point in Ω_1 in $\bar{F}H_1$. Since $|\Omega_1| = |\bar{F} : \bar{F} \cap S| = |\bar{F}H_1 : S|$ and the orders $|\bar{F}|$ and $|H_1|$ are coprime, S contains a conjugate of H_1 ; without loss of generality we assume that $H_1 \leq S$. Any other stabilizer of a point is equal to S^f for $f \in \bar{F} \setminus S$. We claim that

$$S^f \cap H_1 = 1,$$

which is the same as $S \cap H_1^{f^{-1}} = 1$. But all the conjugates H_1^x for $x \in \bar{F}$ are distinct and disjoint and their union contains all the elements of $\bar{F}H_1$ of orders dividing $|H_1|$; the same is true for the conjugates of H_1 in S . Therefore the only conjugates of H_1 intersecting S are H_1^s for $s \in S$.

The H -orbits of elements of regular H_1 -orbits are regular H -orbits. Thus there is one non-regular H -orbit on Ω – the H -orbit of the fixed point of H_1 on Ω_1 . \square

Consider any regular H -orbit on the set Ω , which we temporarily denote by $\{W_h \mid h \in H\}$. Let $X = \bigoplus_{h \in H} W_h$. Then, as before,

$$C_X(H) = \left\{ \sum_{h \in H} xh \mid x \in W_1 \right\}.$$

Since $C_U(H)$, as a subgroup of U , acts on each W_h by scalar multiplications and centralizes $C_X(H)$ by property (V3), we obtain that, in fact, $C_U(H)$ must act trivially on X .

The sum of the W_i over all regular H -orbits is obviously a free $\mathbb{F}_p H$ -module. Since \tilde{V} is also a free $\mathbb{F}_p H$ -module by property (V5), the sum Y of the W_i over

the only one, by Lemma 2.8, possibly remaining non-regular H -orbit must also be a free $\overline{\mathbb{F}}_p H$ -module. Let $Y = \bigoplus_{h \in H} Zh$ for some $\overline{\mathbb{F}}_p H$ -submodule Z . Then, as before,

$$C_Y(H) = \left\{ \sum_{h \in H} yh \mid y \in Z \right\}.$$

The subgroup $C_U(H)$ acts on each W_i by scalar multiplications. Since Y is the sum over an H -orbit and H centralizes $C_U(H)$, all the W_i in the H -orbit are isomorphic $\overline{\mathbb{F}}_p C_U(H)$ -modules. Hence $C_U(H)$ acts by scalar multiplications on the whole Y . Since $C_U(H)$ centralizes $C_Y(H)$ by property (V3), it follows that, in fact, $C_U(H)$ must act trivially on Y .

As a result, $C_U(H)$ acts trivially on W . Since this is true for every factor of the series (2.2) and the order of U is coprime to the characteristic p (or to the order of \tilde{V}), it follows that $C_U(H)$ acts trivially on \tilde{V} , contrary to property (V1). This contradiction completes the proof. \square

3 Bounding the exponent

Here we bound the exponent of a group with a metacyclic Frobenius group of automorphisms. But first we develop the requisite Lie ring technique. The following general definitions are also used in subsequent sections.

The Lie subring (or subalgebra) generated by a subset U is denoted by $\langle U \rangle$, and the ideal generated by U by $\text{id}\langle U \rangle$. Products in a Lie ring are called commutators. We use the term ‘‘span’’ both for the subspace (in the case of algebras) and for the additive subgroup generated by a given subset. For subsets X, Y we denote by $[X, Y]$ the span of all commutators $[x, y]$, $x \in X, y \in Y$; this is an ideal if X, Y are ideals. Terms of the derived series of a Lie ring L are defined as $L^{(0)} = L$, $L^{(k+1)} = [L^{(k)}, L^{(k)}]$. Then L is solvable of derived length at most n (sometimes called ‘‘solvability index’’) if $L^{(n)} = 0$. Terms of the lower central series of L are defined as $\gamma_1(L) = L$, $\gamma_{k+1}(L) = [\gamma_k(L), L]$. Then L is nilpotent of class at most c (sometimes called ‘‘nilpotency index’’) if $\gamma_{c+1}(L) = 0$. We use the standard notation for simple (left-normed) commutators:

$$[x_1, x_2, \dots, x_k] = [\dots[[x_1, x_2], x_3], \dots, x_k]$$

(here the x_i may be elements or subsets).

We use several times the following fact, which helps in proving nilpotency of a solvable Lie ring L : let K be an ideal of a Lie ring L ;

$$\text{if } \gamma_{c+1}(L) \subseteq [K, K] \text{ and } \gamma_{k+1}(K) = 0, \text{ then } \gamma_{c \binom{k+1}{2} - \binom{k}{2} + 1}(L) = 0. \quad (3.1)$$

This can be regarded as a Lie ring analogue of P. Hall’s theorem [6] (with a simpler ‘linear’ proof; see also [24] for the best possible bound for the nilpotency class of L).

Let A be an additively written abelian group. A Lie ring L is A -graded if

$$L = \bigoplus_{a \in A} L_a \quad \text{and} \quad [L_a, L_b] \subseteq L_{a+b}, \quad a, b \in A,$$

where the L_a are subgroups of the additive group of L . Elements of the grading components L_a are called *homogeneous*, and commutators in homogeneous elements *homogeneous commutators*. An additive subgroup H of L is called *homogeneous* if $H = \bigoplus_a (H \cap L_a)$; we then write $H_a = H \cap L_a$. Clearly, any subring or ideal generated by homogeneous additive subgroups is homogeneous. A homogeneous subring and the quotient by a homogeneous ideal can be regarded as A -graded Lie rings with the induced gradings. Also, it is not difficult to see that if H is homogeneous, then so is $C_L(H)$, the centralizer of H , which is, as usual, equal to the set $\{l \in L \mid [l, h] = 0 \text{ for all } h \in H\}$.

An element y of a Lie algebra L is called *ad-nilpotent* if there exists a positive integer n such that $(\text{ad } y)^n = 0$, that is,

$$[x, \underbrace{y, \dots, y}_n] = 0 \quad \text{for all } x \in L.$$

If n is the least integer with this property, then we say that y is *ad-nilpotent of index n* .

Throughout the rest of this section p will denote an arbitrary but fixed prime. Let G be a group. We set

$$D_i = D_i(G) = \prod_{j p^k \geq i} \gamma_j(G)^{p^k}.$$

The subgroups D_i form the *Jennings–Zassenhaus filtration*

$$G = D_1 \geq D_2 \geq \dots$$

of the group G . This series satisfies the inclusions

$$[D_i, D_j] \leq D_{i+j} \quad \text{and} \quad D_i^p \leq D_{pi} \quad \text{for all } i, j.$$

These properties make it possible to construct a Lie algebra $DL(G)$ over \mathbb{F}_p , the field with p elements. Namely, consider the quotients D_i/D_{i+1} as linear spaces over \mathbb{F}_p , and let $DL(G)$ be the direct sum of these spaces. Commutation in the

group G induces a binary operation $[\cdot, \cdot]$ in $DL(G)$. For homogeneous elements $x \in D_i/D_{i+1}$ and $y \in D_j/D_{j+1}$ the operation is defined by

$$[xD_{i+1}, yD_{j+1}] = [x, y]D_{i+j+1} \in D_{i+j}/D_{i+j+1}$$

and extended to arbitrary elements of $DL(G)$ by linearity. It is easy to check that the operation is well-defined and that $DL(G)$ with the operations $+$ and $[\cdot, \cdot]$ is a Lie algebra over \mathbb{F}_p , which is \mathbb{Z} -graded with the D_i/D_{i+1} being the grading components.

For any $x \in D_i \setminus D_{i+1}$ let \bar{x} denote the element xD_{i+1} of $DL(G)$.

Lemma 3.1 (Lazard [15]). *For any $x \in G$ we have $(\text{ad } \bar{x})^p = \text{ad } \bar{x}^p$. Consequently, if x is of finite order p^t , then \bar{x} is ad-nilpotent of index at most p^t .*

Let $L_p(G) = \langle D_1/D_2 \rangle$ be the subalgebra of $DL(G)$ generated by D_1/D_2 ; it is also \mathbb{Z} -graded with grading components $L_i = L_p(G) \cap D_i/D_{i+1}$. The following lemma goes back to Lazard [16]; in the present form it can be found, for example, in [11].

Lemma 3.2. *Suppose that X is a d -generator finite p -group such that the Lie algebra $L_p(X)$ is nilpotent of class c . Then X has a powerful characteristic subgroup of (p, c, d) -bounded index.*

Recall that powerful p -groups were introduced by Lubotzky and Mann in [17]: a finite p -group G is *powerful* if $G^p \geq [G, G]$ for $p \neq 2$ (or $G^4 \geq [G, G]$ for $p = 2$). These groups have many nice properties, so that often a problem becomes much easier once it is reduced to the case of powerful p -groups. The above lemma is quite useful as it allows us to perform such a reduction. We will also require the following lemma. Note that when a \mathbb{Z} -graded Lie algebra L is generated by the component L_1 , that is, $L = \langle L_1 \rangle$, then in fact $L = L_1 \oplus L_2 \oplus \dots$.

Lemma 3.3. *Let $L = \bigoplus L_i$ be a \mathbb{Z} -graded Lie algebra over a field such that $L = \langle L_1 \rangle$ and assume that every homogeneous component L_i is spanned by elements that are ad-nilpotent of index at most r . Suppose further that L is solvable of derived length k and that the component L_1 has finite dimension d . Then L is nilpotent of (d, r, k) -bounded class.*

Proof. Without loss of generality we can assume that $k \geq 2$ and use induction on k . Let M be the last non-trivial term of the derived series of L . By induction we assume that L/M is nilpotent of (d, r, k) -bounded class. In particular, it follows that the dimension of L/M is finite and (d, r, k) -bounded. If $M \leq Z(L)$, then we deduce easily that L is nilpotent of (d, r, k) -bounded class. So assume that M satisfies $M \not\leq Z(L)$. Let j be the biggest index such that the component L_j is not

contained in $C_L(M)$. Thus, $K = L_j + C_L(M)$ is a non-abelian ideal in L . Since the dimension of L/M is finite and (d, r, k) -bounded, there exist boundedly many elements $a_1, \dots, a_m \in L_j$ such that $K = \langle a_1, \dots, a_m, C_L(M) \rangle$ and each of the elements a_1, \dots, a_m is ad-nilpotent of index at most r . Taking into account that j here is (d, r, k) -bounded, we can use backward induction on j . Therefore, by induction $L/[K, K]$ is nilpotent of (d, r, k) -bounded class.

Set $s = m(r - 1) + 1$ and consider the ideal $S = [M, K, \dots, K]$, where K occurs s times. Then S is spanned by commutators of the form $[m, b_1, \dots, b_s]$ where $m \in M$ and b_1, \dots, b_s are not necessarily distinct elements from $\{a_1, \dots, a_m\}$. Since the number s is big enough, it is easy to see that there are r indices i_1, \dots, i_r such that $b_{i_1} = \dots = b_{i_r}$. Taking into account that $K/C_L(M)$ is abelian, we remark that $[m, x, y] = [m, y, x]$ for all $m \in M$ and $x, y \in K$. Therefore we can assume without loss of generality that $b_1 = \dots = b_r$. Since b_1 is ad-nilpotent of index at most r , it follows that $[m, b_1, \dots, b_s] = 0$. Thus we have $S = 0$, which means that M is contained in the s th term of the upper central series of K . The fact that L/M is nilpotent of (d, r, k) -bounded class now implies that K is nilpotent of (d, r, k) -bounded class as well. Combining this with the fact that $L/[K, K]$ is also nilpotent of (d, r, k) -bounded class and using the Lie ring analogue (3.1) of P. Hall's theorem, we deduce that L is also nilpotent of (d, r, k) -bounded class. The proof is complete. \square

We now prove the main result of this section.

Theorem 3.4. *Suppose that a finite Frobenius group FH with cyclic kernel F and complement H acts on a finite group G in such a manner that $C_G(F) = 1$ and $C_G(H)$ has exponent e . Then the exponent of G is bounded solely in terms of e and $|FH|$.*

Proof. By Theorem 2.1, G is solvable. The nilpotent length (Fitting height) of G is bounded in terms of $|F|$ by Dade's theorem [3]. Therefore it is sufficient to bound the exponents of the factors of the Fitting series of G . By Lemma 2.2 and Theorem 2.3 each of them inherits the hypotheses $C_G(F) = 1$ and $C_G(H)^e = 1$. Therefore we can assume from the outset that G is a finite p -group for some prime p . In view of Lemma 2.4, $G = \langle C_G(H)^f \mid f \in F \rangle$, so p divides e . We assume without loss of generality that e is a p -power. Let $x \in G$. It is clear that x is contained in an FH -invariant subgroup of G with at most $|FH|$ generators. Therefore without loss of generality we can assume that G is $|FH|$ -generated.

Any group of automorphisms of the group G acts naturally on every factor of the Jennings–Zassenhaus filtration of G . This action induces an action by automorphisms on the Lie algebra $L = L_p(G)$. Lemma 2.2 shows that F is fixed-point-free on every factor of the Jennings–Zassenhaus filtration. Hence $C_L(F) = 0$.

Kreknin's theorem [13] now tells us that L is solvable of $|F|$ -bounded derived length.

Every homogeneous component L_i of the Lie algebra L can be regarded as an FH -invariant subgroup of the corresponding quotient D_i/D_{i+1} of the Jennings–Zassenhaus series. By Lemma 2.4 we obtain that L_i is generated by the centralizers $C_{L_i}(H)^f$ where f ranges through F . Moreover, Theorem 2.3 implies that every element of $C_{L_i}(H)^f$ is the image of some element of $C_G(H)^f$, the order of which divides e . It follows by Lemma 3.1 that the additive group L_i is generated by elements that are ad-nilpotent of index at most e . We now deduce from Lemma 3.3 that L is nilpotent of $(e, |FH|)$ -bounded class.

Lemma 3.2 now tells us that the group G has a powerful characteristic subgroup of $(e, |FH|)$ -bounded index. It suffices to bound the exponent of this powerful subgroup so we can just assume that G is powerful. Powerful p -groups have the property that if a powerful p -group G is generated by elements of order dividing p^k , then the exponent of G also divides p^k (see [4, Lemma 2.2.5]). Combining this with the fact that our group G is generated by elements of $C_G(H)^f$, $f \in F$, we conclude that G has exponent e . \square

4 Cyclically graded Lie rings with ‘selective nilpotency’ condition

In this section we develop a Lie ring theory which is used in Section 5 below for studying groups G and Lie rings L with a metacyclic Frobenius group of automorphisms FH . This theory is stated in terms Lie rings with finite cyclic grading, which will arise from the ‘eigenspaces’ for F . By Kreknin's theorem ([13]) the condition

$$C_L(F) = 0$$

implies the solvability of L of derived length bounded in terms of $|F|$, but our aim is to obtain nilpotency of class bounded in terms of $|H|$ and the nilpotency class of $C_L(H)$. (Here $C_L(F)$ and $C_L(H)$ denote the fixed-point subrings for F and H .) The nilpotency of $C_L(H)$ of class c implies certain restrictions on the commutation of the grading components, which we nickname ‘selective nilpotency’. For example, in [9] it was shown that if $c = 1$, that is, $C_L(H)$ is abelian, and $|F|$ is a prime, then each component commutes with all but at most $(c, |H|)$ -boundedly many components, which in turn implies a strong bound for the nilpotency class of L . In this section we work with another, rather technical ‘selective nilpotency’ condition. As we shall see in Section 5, this condition actually arises quite naturally in dealing with a Lie ring admitting a metacyclic Frobenius group of automorphisms; similar conditions were exploited earlier in the paper [19] on double Frobenius groups. In this section we virtually achieve a good $(c, |H|)$ -bounded de-

rived length and Engel-type conditions, which prepare ground for the next section, where nilpotency of $(c, |H|)$ -bounded class is finally obtained by using additional considerations.

Numerical preliminaries

The next two lemmas are elementary facts on polynomials.

Lemma 4.1. *Let K be a field of characteristic 0 containing a primitive q th root of unity ω . Suppose that $m = \omega^{i_1} + \dots + \omega^{i_m}$ for some positive integer m and some $0 \leq i_1, \dots, i_m \leq q - 1$. Then $i_1 = \dots = i_m = 0$.*

Proof. Without loss of generality it can be assumed that $K = \mathbb{Q}[\omega]$. Since we have $m = |\omega^{i_1}| + \dots + |\omega^{i_m}|$, the lemma follows from the triangle inequality. \square

Lemma 4.2. *Let $g_1(x) = a_s x^s + \dots + a_0$ and $g_2(x) = b_t x^t + \dots + b_0$ be polynomials with integer coefficients that have no non-zero common complex roots and let $M = \max_{i,j} \{|a_i|, |b_j|\}$. Suppose that n_0 is a positive integer such that g_1 and g_2 have a non-zero common root in $\mathbb{Z}/n_0\mathbb{Z}$. Then*

$$n_0 \leq 2^{2^{(s+t-1)}-1} M^{2^{(s+t-1)}}.$$

Proof. We use induction on $s + t$. If one of the polynomials has degree 0, say, $g_1(x) = b$, then n_0 divides $|b|$ and the statement is correct. Therefore we can assume that s and t are both positive and $a_s, b_t \neq 0$. Let $s \geq t$. We set

$$p(x) = b_t g_1(x) - g_2(x) a_s x^{s-t}.$$

The polynomial $p(x)$ has degree at most $s - 1$ and the polynomials $g_2(x)$ and $p(x)$ have a non-zero common root in $\mathbb{Z}/n_0\mathbb{Z}$ but not in \mathbb{C} . We observe that $p(x)$ is of the form

$$p(x) = c_{s-1} x^{s-1} + \dots + c_0,$$

where

$$\begin{aligned} c_i &= b_t a_i - a_s b_{i-s+t} && \text{for } i = s - t, \dots, s - 1, \\ c_i &= b_t a_i && \text{for } i = 0, \dots, s - t - 1. \end{aligned}$$

Let $M_0 = \max_{i,j} \{|c_i|, |b_j|\}$. By induction, $n_0 \leq 2^{2^{(s+t-2)}-1} M_0^{2^{(s+t-2)}}$. Using that $M_0 \leq 2M^2$ we compute

$$n_0 \leq 2^{2^{(s+t-2)}-1} M_0^{2^{(s+t-2)}} \leq 2^{2^{(s+t-1)}-1} M^{2^{(s+t-1)}},$$

as required. \square

In Section 5 below we shall consider a Frobenius group FH with cyclic kernel $F = \langle f \rangle$ of order n and (necessarily cyclic) complement $H = \langle h \rangle$ of order q acting as $f^h = f^r$ for $1 \leq r \leq n-1$. This is where the following conditions come from for numbers n, q, r , which we fix for the rest of this section:

$$n, q, r \text{ are positive integers such that } 1 \leq r \leq n-1 \text{ and the image of } r \text{ in } \mathbb{Z}/d\mathbb{Z} \text{ is a primitive } q\text{th root of } 1 \text{ for every divisor } d \text{ of } n. \quad (4.1)$$

In particular, q divides $d-1$ for every divisor d of n . When convenient, we freely identify r with its image in $\mathbb{Z}/n\mathbb{Z}$ and regard r as an element of $\mathbb{Z}/n\mathbb{Z}$ such that $r^q = 1$.

Definition. Let a_1, \dots, a_k be not necessarily distinct non-zero elements of $\mathbb{Z}/n\mathbb{Z}$. We say that the sequence (a_1, \dots, a_k) is r -dependent if

$$a_1 + \dots + a_k = r^{\alpha_1} a_1 + \dots + r^{\alpha_k} a_k$$

for some elements $\alpha_i \in \{0, 1, 2, \dots, q-1\}$ not all of which are zero. If the sequence (a_1, \dots, a_k) is not r -dependent, we call it r -independent.

Remark 4.3. A single non-zero element $a \in \mathbb{Z}/n\mathbb{Z}$ is always r -independent: if $a = r^\alpha a$ for $\alpha \in \{1, 2, \dots, q-1\}$, then $a = 0$ by (4.1).

Notation. For a given r -independent sequence (a_1, \dots, a_k) let $D(a_1, \dots, a_k)$ denote the set of all $j \in \mathbb{Z}/n\mathbb{Z}$ such that (a_1, \dots, a_k, j) is r -dependent.

Lemma 4.4. If (a_1, \dots, a_k) is r -independent, then $|D(a_1, \dots, a_k)| \leq q^{k+1}$.

Proof. Suppose that (a_1, \dots, a_k, j) is r -dependent. We have

$$a_1 + a_2 + \dots + a_k + j = r^{i_1} a_1 + r^{i_2} a_2 + \dots + r^{i_k} a_k + r^{i_0} j$$

for suitable $0 \leq i_s \leq q-1$, where not all of the i_s equal to 0. In fact, $i_0 \neq 0$, for otherwise the sequence (a_1, \dots, a_k) would not be r -independent. Moreover, $1 - r^{i_0}$ is invertible because by our assumption (4.1) the image of r in $\mathbb{Z}/d\mathbb{Z}$ is a primitive q th root of 1 for every divisor d of n . We see that

$$j = (r^{i_1} a_1 + \dots + r^{i_k} a_k - a_1 - a_2 - \dots - a_k) / (1 - r^{i_0}),$$

so there are at most q^{k+1} possibilities for j , as required. \square

We will need a sufficient condition for a sequence (a_1, \dots, a_k) to contain an r -independent subsequence.

Lemma 4.5. *Suppose that for some m a sequence (a_1, \dots, a_k) of non-zero elements of $\mathbb{Z}/n\mathbb{Z}$ contains at least $q^m + m$ different values. Then one can choose an r -independent subsequence $(a_1, a_{i_2}, \dots, a_{i_m})$ of m elements that contains a_1 .*

Proof. If $m = 1$, the lemma is obvious, as a_1 is r -independent by Remark 4.3. So we assume that $m \geq 2$ and use induction on m . By induction we can choose an r -independent subsequence of length $m - 1$ starting from a_1 . Without loss of generality we can assume that $(a_1, a_2, \dots, a_{m-1})$ is an r -independent subsequence. By Lemma 4.4 there are at most q^m distinct elements in $D(a_1, a_2, \dots, a_{m-1})$. Since there are at least $q^m + m$ different values in the sequence (a_1, \dots, a_k) , the sequence $a_m, a_{m+1} \dots, a_k$ contains at least $q^m + 1$ different values. So we can always choose an element $b \notin D(a_1, a_2, \dots, a_{m-1})$ among $a_m, a_{m+1} \dots, a_k$ such that the sequence $(a_1, a_2, \dots, a_{m-1}, b)$ is r -independent. \square

‘Selective nilpotency’ condition on graded Lie algebras

To recall the definitions of graded Lie algebras, homogeneous elements and commutators, see the beginning of Section 3. Here we work with a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring L such that $L_0 = 0$. Formally the symbol $\mathbb{Z}/n\mathbb{Z}$ here means the additive group $\mathbb{Z}/n\mathbb{Z}$. However, it will be convenient to use the same symbol to denote also the ring $\mathbb{Z}/n\mathbb{Z}$. To avoid overloaded notation, we adopt the following convention.

Index Convention. Henceforth a small Latin letter with an index $i \in \mathbb{Z}/n\mathbb{Z}$ will denote a homogeneous element in the grading component L_i , with the index only indicating which component this element belongs to: $x_i \in L_i$. We will not be using numbering indices for elements of the L_i , so that different elements can be denoted by the same symbol when it only matters which component the elements belong to. For example, x_i and x_j can be different elements of L_i , so that $[x_i, x_j]$ can be a non-zero element of L_{2i} .

Note that under Index Convention a homogeneous commutator belongs to the component L_s with index s equal to the sum of indices of all the elements involved in this commutator.

Definition. Let n, q, r be integers defined by (4.1). We say that a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring L satisfies the *selective c -nilpotency condition* if, under Index Convention,

$$[x_{d_1}, x_{d_2}, \dots, x_{d_{c+1}}] = 0 \quad \text{whenever } (d_1, \dots, d_{c+1}) \text{ is } r\text{-independent.} \quad (4.2)$$

Notation. We use the usual notation (k, l) for the greatest common divisor of integers k, l .

Given $b \in \mathbb{Z}/n\mathbb{Z}$, we denote by $o(b)$ the order of b (in the additive group). Thus, $o(b)$ is the least positive integer such that

$$\underbrace{b + b + \dots + b}_{o(b)} = 0.$$

In the following lemmas the condition $L_0 = 0$ ensures that all indices of non-trivial elements are non-zero, so all sequences of such indices are either r -dependent or r -independent.

Lemma 4.6. *Suppose that a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring L with $L_0 = 0$ satisfies the selective c -nilpotency condition (4.2), and let b be an element of $\mathbb{Z}/n\mathbb{Z}$ such that*

$$o(b) > 2^{2^{2q-3}-1} c^{2^{2q-3}}.$$

Then there are at most q^{c+1} elements $a \in \mathbb{Z}/n\mathbb{Z}$ such that

$$[L_a, \underbrace{L_b, \dots, L_b}_c] \neq 0.$$

Proof. Suppose that

$$[L_a, \underbrace{L_b, \dots, L_b}_c] \neq 0.$$

By (4.2) the sequence

$$(a, \underbrace{b, \dots, b}_c)$$

must be r -dependent and so we have $a + b + \dots + b = r^{i_0}a + r^{i_1}b + \dots + r^{i_c}b$ for suitable $0 \leq i_j \leq q - 1$, where at least once $i_j \neq 0$. First suppose that we have $i_0 = 0$. Then for some $1 \leq m \leq c$ and some $1 \leq j_1, \dots, j_m \leq q - 1$ we have $(r^{j_1} + \dots + r^{j_m} - m)b = 0$, where the j_s are not necessarily different. Put $m_0 = r^{j_1} + \dots + r^{j_m} - m$ and $n_0 = (n, m_0)$. Thus, $o(b)$ divides n_0 and $r^{j_1} + \dots + r^{j_m} - m = 0$ in $\mathbb{Z}/n_0\mathbb{Z}$. We collect terms re-writing $r^{j_1} + \dots + r^{j_m} - m$ as $b_{k_1}r^{k_1} + \dots + b_{k_l}r^{k_l} - m$, where $b_{k_i} > 0$, $\sum_{i=1}^l b_{k_i} = m$, and $q > k_1 > k_2 > \dots > k_l > 0$ are all different. By Lemma 4.1 the polynomials $X^{q-1} + \dots + X + 1$ and $b_{k_1}X^{k_1} + \dots + b_{k_l}X^{k_l} - m$ have no common complex roots. On the other hand, these polynomials have a common non-zero root in $\mathbb{Z}/n_0\mathbb{Z}$: namely, the image of r . Therefore, by Lemma 4.2,

$$n_0 \leq 2^{2^{2q-3}-1} m^{2^{2q-3}} \leq 2^{2^{2q-3}-1} c^{2^{2q-3}}.$$

This yields a contradiction since $n_0 \geq o(b) > 2^{2^{2q-3}-1} c^{2^{2q-3}}$.

Hence, $i_0 \neq 0$. In this case $1 - r^{i_0}$ is invertible by assumption (4.1). Therefore,

$$a = (r^{i_1}b + \dots + r^{i_c}b - cb)/(1 - r^{i_0}),$$

so there are at most q^{c+1} possibilities for a . The lemma follows. □

Lemma 4.7. *Suppose that a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring L with $L_0 = 0$ satisfies the selective c -nilpotency condition (4.2). There is a (c, q) -bounded number w such that*

$$[L, \underbrace{L_b, \dots, L_b}_w] = 0$$

whenever b is an element of $\mathbb{Z}/n\mathbb{Z}$ such that $o(b) > \max\{2^{2^{2q-3}-1}c^{2^{2q-3}}, q^{c+1}\}$.

Proof. Denote by $N(b)$ the set of all $a \in \mathbb{Z}/n\mathbb{Z}$ such that

$$[L_a, \underbrace{L_b, \dots, L_b}_c] \neq 0.$$

By Lemma 4.6, $N = |N(b)| \leq q^{c+1}$. If for some $t \geq 1$ we have

$$[L_a, \underbrace{L_b, \dots, L_b}_{c+t}] \neq 0,$$

then all elements $a, a + b, a + 2b, a + tb$ belong to $N(b)$. It follows that either

$$[L_a, \underbrace{L_b, \dots, L_b}_{c+N}] = 0$$

or $sb = 0$ for some $s \leq N$. But the latter case with $o(b) \leq N$ is impossible by the hypothesis $o(b) > q^{c+1} \geq N$. Hence,

$$[L, \underbrace{L_b, \dots, L_b}_{c+q^{c+1}}] = 0. \quad \square$$

Now recall that for a given r -independent sequence (a_1, \dots, a_k) we denote by $D(a_1, \dots, a_k)$ the set of all $j \in \mathbb{Z}/n\mathbb{Z}$ such that (a_1, \dots, a_k, j) is r -dependent.

Let (d_1, \dots, d_c) be an arbitrary r -independent sequence, which we consider as fixed in the next few lemmas.

Lemma 4.8. *Suppose that a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring L with $L_0 = 0$ satisfies the selective c -nilpotency condition (4.2), and let $U = [u_{d_1}, \dots, u_{d_c}]$ be a homogeneous commutator with the r -independent sequence of indices (d_1, \dots, d_c) (under*

Index Convention). Then every commutator of the form

$$[U, x_{i_1}, \dots, x_{i_t}] \quad (4.3)$$

can be written as a linear combination of commutators of the form

$$[U, m_{j_1}, \dots, m_{j_s}], \quad (4.4)$$

where $j_k \in D(d_1, \dots, d_c)$ and $s \leq t$. The case $s = t$ is possible only if we have $i_k \in D(d_1, \dots, d_c)$ for all $k = 1, \dots, t$.

Proof. The assertion is obviously true for $t = 0$. Let $t = 1$. If $i_1 \in D(d_1, \dots, d_c)$, then $[U, x_{i_1}]$ is of the required form. If $i_1 \notin D(d_1, \dots, d_c)$, then $[U, x_{i_1}] = 0$ by (4.2) and there is nothing to prove.

Let us assume that $t > 1$ and use induction on t . If all the indices i_j belong to $D(d_1, \dots, d_c)$, then the commutator $[U, x_{i_1}, \dots, x_{i_t}]$ is of the required form with $s = t$. Suppose that in (4.3) there is an element x_{i_k} with the index i_k that does not belong to $D(d_1, \dots, d_c)$. We choose such an element with k as small as possible and use k as a second induction parameter.

If $k = 1$, then the commutator (4.3) is zero and we are done. Thus, suppose that $k \geq 2$ and write

$$\begin{aligned} [U, \dots, x_{i_{k-1}}, x_{i_k}, \dots, x_{i_t}] &= [U, \dots, x_{i_k}, x_{i_{k-1}}, \dots, x_{i_t}] \\ &\quad + [U, \dots, [x_{i_{k-1}}, x_{i_k}], \dots, x_{i_t}]. \end{aligned}$$

By induction hypothesis, the commutator $[U, \dots, [x_{i_{k-1}}, x_{i_k}], \dots, x_{i_t}]$ is a linear combination of commutators of the form (4.4) because it is shorter than (4.3), while the commutator $[U, \dots, x_{i_k}, x_{i_{k-1}}, \dots, x_{i_t}]$ is a linear combination of commutators of the form (4.4) because the index that does not belong to $D(d_1, \dots, d_c)$ here occurs closer to U than in (4.3). \square

Corollary 4.9. *Let L and U be as in Lemma 4.8. Then the ideal of L generated by U is spanned by commutators of the form (4.4).*

In the next lemma we obtain a rather detailed information about the ideal generated by U . Basically it says that $\text{id}\langle U \rangle$ is generated by commutators of the form (4.4) in which right after U there are boundedly many elements with indices in $D(d_1, \dots, d_c)$ followed only by elements with indices of small additive order in $\mathbb{Z}/n\mathbb{Z}$.

Notation. From now on we fix the (c, q) -bounded number

$$N(c, q) = \max\{2^{2q-3}-1c^{2^{2q-3}}, q^{c+1}\},$$

which appears in Lemma 4.7.

For our fixed r -independent sequence (d_1, \dots, d_c) , we denote by A the set of all elements $b \in D(d_1, \dots, d_c)$ such that $o(b) > N(c, q)$ and by B the set of all elements $b \in D(d_1, \dots, d_c)$ such that $o(b) \leq N(c, q)$. Let $D = |D(d_1, \dots, d_c)|$ and let w be the number given by Lemma 4.7.

Lemma 4.10. *Let L and U be as in Lemma 4.8. The ideal of L generated by U is spanned by commutators of the form*

$$[U, m_{i_1}, \dots, m_{i_u}, m_{i_{u+1}}, \dots, m_{i_v}], \tag{4.5}$$

where $u \leq (w - 1)D$, with $i_k \in D(d_1, \dots, d_c)$ for $k \leq u$, and $i_k \in B$ for $k > u$.

Proof. Let R be the span of all commutators of the form (4.5). It is sufficient to show that every commutator $W = [U, m_{j_1}, \dots, m_{j_s}]$ of the form (4.4) belongs to R . If $s \leq (w - 1)D$, it is clear that $W \in R$, so we assume that $s > (w - 1)D$ and use induction on s . Write

$$W = [U, m_{j_1}, \dots, m_{j_t}, m_{j_{t-1}}, \dots, m_{j_s}] + [U, m_{j_1}, \dots, [m_{j_{t-1}}, m_{j_t}], \dots, m_{j_s}]. \tag{4.6}$$

If $j_{t-1} + j_t \in D(d_1, \dots, d_c)$, then the second summand is of the form (4.4). Since it is shorter than W , it follows that it belongs to R by the induction hypothesis. If $j_{t-1} + j_t \notin D(d_1, \dots, d_c)$, then by Lemma 4.8 the second summand is a linear combination of commutators of the form (4.4) each of which is shorter than W .

Thus, in either case the second summand in (4.6) belongs to R . It follows that the commutator W does not change modulo R under any permutation of the m_{j_k} . If among the m_{j_k} there are at least w elements with the same index $j_k \in A$, we move these elements next to each other. Then it follows from Lemma 4.7 that $W = 0$. Suppose now that all the indices $j_k \in A$ occur less than w times. We place all these elements right after the U . This initial segment has length at most $D(w - 1) + c$, so the resulting commutator takes the required form (4.5). \square

Corollary 4.11. *Suppose that a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring L with $L_0 = 0$ satisfies the selective c -nilpotency condition (4.2), and let (d_1, \dots, d_c) be an r -independent sequence. Then the ideal $\text{id}\{[L_{d_1}, \dots, L_{d_c}]\}$ has (c, q) -boundedly many non-trivial components of the induced grading.*

Proof. Let $U = [u_{d_1}, \dots, u_{d_c}]$ be an arbitrary homogeneous commutator with the given indices (under Index Convention). Since in (4.5) we have $i_k \in D(d_1, \dots, d_c)$ for all $k = 1, \dots, u$, the sum of all indices of the initial segment $[U, m_{i_1}, \dots, m_{i_u}]$ can take at most D^u values. Denote by Y the order of $\langle B \rangle$, the subgroup of $\mathbb{Z}/n\mathbb{Z}$ generated by B . Clearly, the sum of the remaining indices in (4.5) belongs to $\langle B \rangle$.

It follows that the sum of all indices in (4.5) can take at most $D^u Y$ values. As the order of every element in B is (c, q) -bounded and there are only (c, q) -boundedly many elements in B , it follows that Y is (c, q) -bounded. By Lemmas 4.4, 4.7, and 4.10 the number u is also (c, q) -bounded. So $\text{id}\langle U \rangle$ has (c, q) -boundedly many non-trivial components.

It follows from the proofs of Lemmas 4.8 and 4.10 that the set of indices of all possible non-trivial components in $\text{id}\langle U \rangle$ is completely determined by the tuple (d_1, \dots, d_c) and does not depend on the choice of $U = [u_{d_1}, \dots, u_{d_c}]$. Since the ideal $\text{id}\langle [L_{d_1}, \dots, L_{d_c}] \rangle$ is the sum of ideals $\text{id}\langle [u_{d_1}, \dots, u_{d_c}] \rangle$ over all possible u_{d_1}, \dots, u_{d_c} , the result follows. \square

Lemma 4.12. *Suppose that a homogeneous ideal T of a Lie ring L has only e non-trivial components. Then L has at most e^2 components that do not centralize T .*

Proof. Let T_{i_1}, \dots, T_{i_e} be the non-trivial homogeneous components of T and let $S = \{i_1, \dots, i_e\}$. Suppose that L_i does not centralize T . Then for some $j \in S$ we have $i + j \in S$. So there are at most $|S| \times |S|$ possibilities for i , as required. \square

Proposition 4.13. *Suppose that a $(\mathbb{Z}/n\mathbb{Z})$ -graded Lie ring L with $L_0 = 0$ satisfies the selective c -nilpotency condition (4.2). Then L is solvable of (c, q) -bounded derived length $f(c, q)$.*

Note that L is solvable of n -bounded derived length by Kreknin’s theorem, but we need a bound for the derived length in terms of c and q .

Proof. We use induction on c . If $c = 0$, then $L = 0$ and there is nothing to prove. Indeed, $L_0 = 0$ by hypothesis, and for $d \neq 0$ we have $L_d = 0$ by (4.2), since any element $d \neq 0$ is r -independent by Remark 4.3.

Now let $c \geq 1$. Let I be the ideal of the given Lie ring L generated by all commutators $[L_{i_1}, \dots, L_{i_c}]$, where (i_1, \dots, i_c) ranges through all r -independent sequences of length c . The induced $(\mathbb{Z}/n\mathbb{Z})$ -grading of L/I has trivial zero-component and L/I satisfies the selective $(c - 1)$ -nilpotency condition. By the induction hypothesis L/I is solvable of bounded derived length, say, f_0 , that is, $L^{(f_0)} \leq I$.

Consider an arbitrary r -independent sequence (i_1, \dots, i_c) and the ideal

$$T = \text{id}\langle [L_{i_1}, \dots, L_{i_c}] \rangle.$$

We know from Corollary 4.11 that there are only (c, q) -boundedly many, say, e , non-trivial grading components in T . By Lemma 4.12 there are at most e^2 components that do not centralize T . Since $C_L(T)$ is also a homogeneous ideal, it follows that the quotient $L/C_L(T)$ has at most e^2 non-trivial components. Since the induced $(\mathbb{Z}/n\mathbb{Z})$ -grading of $L/C_L(T)$ also has trivial zero component, by Shalev’s

generalization [23] of Kreknin’s theorem we conclude that $L/C_L(T)$ is solvable of e -bounded derived length, say, f_1 . Therefore $L^{(f_1)}$, the corresponding term of the derived series, centralizes T . Since f_1 does not depend on the choice of the r -independent tuple (i_1, \dots, i_c) and I is the sum of all such ideals T , it follows that $[L^{(f_1)}, I] = 0$. Recall that $L^{(f_0)} \leq I$. Hence, $[L^{(f_1)}, L^{(f_0)}] = 0$. Thus L is solvable of (c, q) -bounded derived length at most $\max\{f_0, f_1\} + 1$. \square

Combinatorial corollary

We now state a combinatorial corollary of Lemma 4.7 and Proposition 4.13 that we shall need in the next section for dealing with non-semisimple automorphisms, when eigenspaces do not form a direct sum.

We use the following notation:

$$\delta_1 = [x_1, x_2], \quad \delta_{k+1} = [\delta_k(x_1, \dots, x_{2^k}), \delta_k(x_{2^k-1}, \dots, x_{2^{k+1}})].$$

Corollary 4.14. *Let n, q, r be positive integers such that $1 \leq r \leq n - 1$ and the image of r in $\mathbb{Z}/d\mathbb{Z}$ is a primitive q th root of 1 for every divisor d of n .*

- (a) *For the function $f(c, q)$ given by Proposition 4.13 above, the following holds. If we arbitrarily and formally assign lower indices $i_1, i_2, \dots \in \mathbb{Z}$ to elements y_{i_1}, y_{i_2}, \dots of an arbitrary Lie ring, then the commutator*

$$\delta_{f(c,q)}(y_{i_1}, y_{i_2}, \dots, y_{i_{2^{f(c,q)}}})$$

can be represented as a linear combination of commutators in the same elements $y_{i_1}, y_{i_2}, \dots, y_{i_{2^{f(c,q)}}$ each of which contains either a subcommutator with zero modulo n sum of indices or a subcommutator of the form

$$[g_{u_1}, g_{u_2}, \dots, g_{u_{c+1}}]$$

with an r -independent sequence (u_1, \dots, u_{c+1}) of indices, where elements g_j are commutators in $y_{i_1}, y_{i_2}, \dots, y_{i_{2^{f(c,q)}}$ such that the sum of indices of all the elements involved in g_j is congruent to j modulo n .

- (b) *For the function $w = w(c, q)$ given by Lemma 4.7, any commutator*

$$[y_a, \underbrace{x_b, y_b, \dots, z_b}_w],$$

with w elements with the same index b over the brace, can be represented as a linear combination of the same form as in (a) whenever the image of b in $\mathbb{Z}/n\mathbb{Z}$ is such that $o(b) > N(c, q) = \max\{2^{2^{q-3}} - 1, c^{2^{2q-3}}, q^{c+1}\}$.

Proof. (a) Let M be a free Lie ring freely generated by $y_{i_1}, y_{i_2}, \dots, y_{i_{2f(c,q)}}$. We define for each $i = 0, 1, \dots, n - 1$ the additive subgroup M_i of M generated (in the additive group) by all commutators in the generators y_{i_j} with the sum of indices congruent to i modulo n . Then, obviously, $M = M_0 \oplus M_1 \oplus \dots \oplus M_{n-1}$ and $[M_i, M_j] \subseteq M_{i+j \pmod n}$, so that this is a $(\mathbb{Z}/n\mathbb{Z})$ -grading. By Proposition 4.13 we obtain

$$\delta_{f(c,q)}(y_{i_1}, y_{i_2}, \dots, y_{i_{2f(c,q)}}) \in \text{id}\langle M_0 \rangle + \sum_{\substack{(u_1, \dots, u_{c+1}) \\ \text{is } r\text{-independent}}} \text{id}\langle [M_{u_1}, M_{u_2}, \dots, M_{u_{c+1}}] \rangle.$$

By the definition of M_i this inclusion is equivalent to the required equality in the conclusion of Corollary 4.14 (a). Since the elements $y_{i_1}, y_{i_2}, \dots, y_{i_{2f(c,q)}}$ freely generate the Lie ring M , the same equality holds in any Lie ring.

(b) The proof of the second statement is obtained by the same arguments with the only difference that Lemma 4.7 is applied instead of Proposition 4.13. \square

5 Bounding the nilpotency class

In this section we obtain bounds for the nilpotency class of groups and Lie rings admitting a metacyclic Frobenius group of automorphisms with fixed-point-free kernel. Earlier such results were obtained by the second and third authors [19] in the case of the kernel of prime order. Examples at the end of the section show that such nilpotency results are no longer true for non-metacyclic Frobenius groups of automorphisms. The results for groups are consequences of the corresponding results for Lie rings and algebras, by various Lie ring methods.

Lie algebras

We begin with the theorem for Lie algebras, which is devoid of technical details required for arbitrary Lie rings, so that the ideas of proof are more clear. Recall that $C_L(A)$ denotes the fixed-point subalgebra for a group of automorphisms A of a Lie algebra L , and that nilpotency class is also known as nilpotency index.

Theorem 5.1. *Let FH be a Frobenius group with cyclic kernel F of order n and complement H of order q . Suppose that FH acts by automorphisms on a Lie algebra L in such a way that $C_L(F) = 0$ and $C_L(H)$ is nilpotent of class c . Then L is nilpotent of (c, q) -bounded class.*

Note that the assumption of the kernel F being cyclic is essential: see the corresponding examples at the end of the section.

Proof. Our first aim is to reduce the problem to the study of $\mathbb{Z}/n\mathbb{Z}$ -graded Lie algebras satisfying the selective c -nilpotency condition (4.2). Then the solubility of L of (c, q) -bounded derived length will immediately follow from Proposition 4.13. Further arguments are then applied to obtain nilpotency.

Let ω be a primitive n th root of 1. We extend the ground field by ω and denote the resulting Lie algebra by \widetilde{L} . The group FH acts in a natural way on \widetilde{L} and this action inherits the conditions that $C_{\widetilde{L}}(F) = 0$ and $C_{\widetilde{L}}(H)$ is nilpotent of class c . Thus, we can assume that $L = \widetilde{L}$ and the ground field contains ω , a primitive n th root of 1.

We shall need a simple remark that we can assume that $n = |F|$ is not divisible by the characteristic of the ground field. Indeed, if the characteristic p divides n , then the Hall p' -subgroup $\langle \chi \rangle$ of F acts fixed-point-freely on L – otherwise the Sylow p -subgroup $\langle \psi \rangle$ of F would have non-trivial fixed points on the ψ -invariant subspace $C_L(\chi)$, and these would be non-trivial fixed points for F . Thus, L admits the Frobenius group of automorphisms $\langle \chi \rangle H$ with $C_L(\chi) = 0$. Replacing F by $\langle \chi \rangle$ we can assume that p does not divide n .

Let φ be a generator of F . For each $i = 0, \dots, n - 1$ we denote by

$$L_i = \{x \in L \mid x^\varphi = \omega^i x\}$$

the eigenspace for the eigenvalue ω^i . Then

$$[L_i, L_j] \subseteq L_{i+j \pmod{n}} \quad \text{and} \quad L = \bigoplus_{i=0}^{n-1} L_i,$$

so this is a $(\mathbb{Z}/n\mathbb{Z})$ -grading. We also have $L_0 = C_L(F) = 0$.

Since F is cyclic of order n , H is also cyclic. Let $H = \langle h \rangle$ and let $\varphi^{h^{-1}} = \varphi^r$ for some $1 \leq r \leq n - 1$. Then r is a primitive q th root of 1 in $\mathbb{Z}/n\mathbb{Z}$, and, moreover, the image of r in $\mathbb{Z}/d\mathbb{Z}$ is a primitive q th root of 1 for every divisor d of n , since h acts fixed-point-freely on every subgroup of F . Thus, n, q, r satisfy condition (4.1).

The group H permutes the components L_i so that $L_i^h = L_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Indeed, if $x_i \in L_i$, then

$$(x_i^h)^\varphi = x_i^{h\varphi h^{-1}h} = (x_i^{\varphi^r})^h = \omega^{ir} x_i^h.$$

Given $u_k \in L_k$, we temporarily denote $u_k^{h^i}$ by $u_{r^i k}$ (under Index Convention). The sum over any H -orbit belongs to $C_L(H)$ and therefore

$$u_k + u_{rk} + \dots + u_{r^{q-1}k} \in C_L(H).$$

Let $x_{a_1}, \dots, x_{a_{c+1}}$ be homogeneous elements in $L_{a_1}, \dots, L_{a_{c+1}}$, respectively. Consider the elements

$$\begin{aligned} X_1 &= x_{a_1} + x_{ra_1} + \dots + x_{r^{q-1}a_1}, \\ &\vdots \\ X_{c+1} &= x_{a_{c+1}} + x_{ra_{c+1}} + \dots + x_{r^{q-1}a_{c+1}}. \end{aligned}$$

Since all of them lie in $C_L(H)$, which is nilpotent of class c , it follows that

$$[X_1, \dots, X_{c+1}] = 0.$$

After expanding the expressions for the X_i , we obtain on the left a linear combination of commutators in the elements $x_{r^j a_i}$, which in particular involves the term $[x_{a_1}, \dots, x_{a_{c+1}}]$. Suppose that the commutator $[x_{a_1}, \dots, x_{a_{c+1}}]$ is non-zero. Then there must be other terms in the expanded expression that belong to the same component $L_{a_1+\dots+a_{c+1}}$. In other words, then

$$a_1 + \dots + a_{c+1} = r^{\alpha_1} a_1 + \dots + r^{\alpha_{c+1}} a_{c+1}$$

for some $\alpha_i \in \{0, 1, 2, \dots, q-1\}$ not all of which are zeros, so that (a_1, \dots, a_{c+1}) is an r -dependent sequence. This means that L satisfies the selective c -nilpotency condition (4.2). By Proposition 4.13 we obtain that L is solvable of (c, q) -bounded derived length.

We now use induction on the derived length to prove that L is nilpotent of (c, q) -bounded class. If L is abelian, there is nothing to prove. Assume that L is metabelian – this is, in fact, the main part of the proof. When L is metabelian, we have $[x, y, z] = [x, z, y]$ for every $x \in [L, L]$ and $y, z \in L$. The key step is a stronger version of Lemma 4.7 for the metabelian case, without the arithmetical condition on the additive orders.

Lemma 5.2. *Let L be metabelian. There is a (c, q) -bounded number m such that*

$$[L, \underbrace{L_b, \dots, L_b}_m] = 0 \quad \text{for every } b \in \mathbb{Z}/n\mathbb{Z}.$$

Proof. For each $a \in \mathbb{Z}/n\mathbb{Z}$ we denote $[L, L] \cap L_a$ by $[L, L]_a$. Clearly, it suffices to show that

$$[[L, L]_a, \underbrace{L_b, \dots, L_b}_{m-1}] = 0 \quad \text{for every } a, b \in \mathbb{Z}/n\mathbb{Z};$$

we can of course assume that $a, b \neq 0$. First suppose that $(n, b) = 1$. If n is large enough, $n > N(c, q)$, then the order $o(b) = n$ is also large enough and the result

follows by Lemma 4.7 with $m - 1 = w$, where w is given by Lemma 4.7. If, however, $n \leq N(c, q)$, then we find a positive integer $k < n$ such that $a + kb = 0$ in $\mathbb{Z}/n\mathbb{Z}$. (As usual, we freely switch from considering positive integers to their images in $\mathbb{Z}/n\mathbb{Z}$ without changing notation.) Then

$$[L_a, \underbrace{L_b, \dots, L_b}_k] \subseteq L_0 = 0,$$

so that we can put $m - 1 = N(c, q)$.

Now suppose that $(n, b) \neq 1$, and let s be a prime dividing both n and b . If s also divides a , then the result follows by induction on $|F|$ applied to the Lie algebra $C_L(\varphi^{n/s}) = \sum_i L_{si}$ containing both L_a and L_b : this Lie algebra is FH -invariant and φ acts on it as an automorphism of order n/s without non-trivial fixed points. The basis of this induction is the case of $n = |F|$ being a prime, where necessarily $(n, b) = 1$, which has already been dealt with above. (Alternatively, we can refer to the main result of [19], where Theorem 5.1 was proved for F of prime order.) Note that the function $m = m(c, q)$ remains the same in the induction step, so no dependence on $|F|$ arises.

Thus, it remains to consider the case where the prime s divides both b and n and does not divide a . Using the same notation $x_{r^i k} = x_k^{h^i}$ (under Index Convention), we have

$$\left[u_a + u_{ra} + \dots + u_{r^{q-1}a}, \right. \\ \left. \underbrace{(v_b + v_{rb} + \dots + v_{r^{q-1}b}), \dots, (w_b + w_{rb} + \dots + w_{r^{q-1}b})}_c \right] = 0$$

for any c elements $v_b, \dots, w_b \in L_b$, because here sums are elements of $C_L(H)$.

By (4.1) any two indices $r^i a, r^j a$ here are different modulo s , while all the indices above the brace are divisible by s , which divides n . Hence we also have

$$\left[u_a, \underbrace{(v_b + v_{rb} + \dots + v_{r^{q-1}b}), \dots, (w_b + w_{rb} + \dots + w_{r^{q-1}b})}_c \right] = 0. \tag{5.1}$$

Let Z denote the span of all the sums $x_b + x_{rb} + \dots + x_{r^{q-1}b}$ over $x_b \in L_b$ (in fact, Z is the fixed-point subspace of H on $\bigoplus_{i=0}^{q-1} L_{r^i b}$). Then (5.1) means that

$$\left[L_a, \underbrace{Z, \dots, Z}_c \right] = 0.$$

Applying φ^j we also obtain

$$\left[L_a, \underbrace{Z^{\varphi^j}, \dots, Z^{\varphi^j}}_c \right] = \left[L_a^{\varphi^j}, \underbrace{Z^{\varphi^j}, \dots, Z^{\varphi^j}}_c \right] = 0. \tag{5.2}$$

A Vandermonde-type linear algebra argument shows that $L_b \subseteq \sum_{j=0}^{q-1} Z\varphi^j$. We prove this fact in a greater generality, which will be needed later.

Lemma 5.3. *Let $\langle \varphi \rangle$ be a cyclic group of order n , and let ω be a primitive n th root of unity. Suppose that M is a $\mathbb{Z}[\omega]\langle \varphi \rangle$ -module such that $M = \sum_{i=1}^m M_{k_i}$, where $x\varphi = \omega^{k_i}x$ for $x \in M_{k_i}$ and $0 \leq k_1 < k_2 < \dots < k_m < n$. If*

$$z = y_{k_1} + y_{k_2} + \dots + y_{k_m} \quad \text{for } y_{k_i} \in M_{k_i},$$

then for some m -bounded number l_0 every element $n^{l_0}y_{k_s}$ is a $\mathbb{Z}[\omega]$ -linear combination of the elements $z, z\varphi, \dots, z\varphi^{m-1}$.

Proof. We have

$$z\varphi^j = \omega^{jk_1}y_{k_1} + \omega^{jk_2}y_{k_2} + \dots + \omega^{jk_m}y_{k_m}.$$

Giving values $j = 0, \dots, m - 1$, we obtain m linear combinations of the elements $y_{k_1}, y_{k_2}, \dots, y_{k_m}$. Then for every $s = 1, \dots, m$ a suitable linear combination of these linear combinations produces Dy_{k_s} , where D is the Vandermonde determinant of the $m \times m$ matrix of coefficients of these linear combinations, which is equal to

$$\prod_{1 \leq i < j \leq m} (\omega^{k_i} - \omega^{k_j}).$$

Each factor is a product of an (invertible) power of ω and an element $1 - \omega^{k_i - k_j}$. By hypothesis the exponent $k_i - k_j$ is not divisible by n , so $\rho = \omega^{k_i - k_j}$ is a root of the polynomial $1 + x + \dots + x^{n-1} = (x - \rho)g(x)$. By substituting $x = 1$ we obtain that $n = (1 - \rho)g(1)$, so by multiplying Dy_{k_s} by an invertible power of ω and several elements of the type $g(1)$ we obtain $n^{l_0}y_{k_s}$ for some m -bounded number l_0 . As a result, $n^{l_0}y_{k_s}$ is expressed as a $\mathbb{Z}[\omega]$ -linear combination of the elements $z, z\varphi, \dots, z\varphi^{m-1}$. □

In the proof of Lemma 5.2 we can apply Lemma 5.3 with

$$M = L_b + L_{rb} + \dots + L_{r^{q-1}b} \quad \text{and} \quad m = q$$

to $w = v_b + v_{rb} + \dots + v_{r^{q-1}b} \in Z$ for any $v_b \in L_b$, because here all the indices $r^i b$ can be regarded as pairwise distinct residues modulo n by condition (4.1) (that r is a primitive q th root of unity modulo any divisor of n). Using the fact that n is invertible in our ground field, we obtain that $L_b \subseteq \sum_{j=0}^{q-1} Z\varphi^j$.

We now claim that

$$[[L, L]_a, \underbrace{L_b, \dots, L_b}_{q(c-1)+1}] = 0.$$

Indeed, after replacing L_b with $\sum_{j=0}^{q-1} Z^{\varphi^j}$ and expanding the sums, in each commutator of the resulting linear combination we can freely permute the entries Z^{φ^j} , since L is metabelian. Since there are sufficiently many of them, we can place at least c of the same $Z^{\varphi^{j_0}}$ for some j_0 right after $[L, L]_a$ at the beginning, which gives 0 by (5.2). □

We now return to the case of metabelian L in the proof of Theorem 5.1. Let $m = m(c, q)$ be as in Lemma 5.2 and put $g = (m - 1)(q^{c+1} + c) + 2$. For any sequence of g non-zero elements (a_1, \dots, a_g) in $\mathbb{Z}/n\mathbb{Z}$ consider the commutator $[[L, L]_{a_1}, L_{a_2}, \dots, L_{a_g}]$. If the sequence (a_1, \dots, a_g) contains an r -independent subsequence of length $c + 1$ that starts with a_1 , by permuting the L_{a_i} we can assume that a_1, \dots, a_{c+1} is the r -independent subsequence. Then

$$[[L, L]_{a_1}, L_{a_2}, \dots, L_{a_g}] = 0$$

by (4.2). If the sequence (a_1, \dots, a_g) does not contain an r -independent subsequence of length $c + 1$ starting with the element a_1 , then by Lemma 4.5 the sequence (a_1, \dots, a_g) contains at most $q^{c+1} + c$ different values. The number g was chosen big enough to guarantee that either the value of a_1 occurs in (a_1, \dots, a_g) at least $m + 1$ times or, else, another value, different from a_1 , occurs at least m times. Using that $[x, y, z] = [x, z, y]$ for $x \in [L, L]$ we can assume that

$$a_2 = \dots = a_{m+1},$$

in which case it follows from Lemma 5.2 that

$$[[L, L]_{a_1}, L_{a_2}, \dots, L_{a_g}] = 0.$$

Thus, we conclude that L is nilpotent of class at most g .

Now suppose that the derived length of L is at least 3. By induction hypothesis, $[L, L]$ is nilpotent of bounded class. According to the previous paragraph, the quotient $L/[L, L], [L, L]$ is nilpotent of bounded class, as well. Together, this gives nilpotency of L of bounded class by the Lie ring analogue (3.1) of P. Hall’s theorem. □

Lie groups and torsion-free locally nilpotent groups

We now derive the group-theoretic consequences of Theorem 5.1; but the theorem on finite groups will have to wait until we prove a similar result for Lie rings. By the well-known connection between Lie groups and their Lie algebras, the following theorem is an immediate consequence of Theorem 5.1. Recall that $C_G(A)$ denotes the fixed-point subgroup for a group of automorphisms A of a group G , and that nilpotency class is also known as nilpotency index.

Theorem 5.4. *Suppose that a connected Lie group G (complex or real) admits a finite Frobenius group of automorphisms FH with cyclic kernel F of order n and complement H of order q such that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class c . Then G is nilpotent of (c, q) -bounded class.*

Proof. Every automorphism α of G induces the automorphism $d_e\alpha$ of the tangent Lie algebra \mathfrak{g} of G which is the differential of α at identity. The fixed-point subalgebra $C_{\mathfrak{g}}(d_e\alpha)$ is the tangent Lie algebra of the fixed-point subgroup $C_G(\alpha)$ (see, for example, [21, Theorem 3.7]). Therefore the group of automorphisms \overline{FH} of \mathfrak{g} induced by FH has the properties that $C_{\mathfrak{g}}(\overline{F}) = 0$ and $C_{\mathfrak{g}}(\overline{H})$ is nilpotent of class c . (This can also be shown by using the Exp and Log functors, which locally commute with automorphisms of G .) By Theorem 5.1, the Lie algebra \mathfrak{g} is nilpotent of (c, q) -bounded class. Since our Lie group G is connected, this implies the same result for G . \square

Another corollary of Theorem 5.1 follows by a similar Lie ring method – the Mal'cev correspondence, which is also based on the Exp and Log functors and the Baker–Campbell–Hausdorff formula.

Theorem 5.5. *Suppose that a locally nilpotent torsion-free group G admits a finite Frobenius group of automorphisms FH with cyclic kernel F of order n and complement H of order q such that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class c . Then G is nilpotent of (c, q) -bounded class.*

Proof. It is clearly sufficient to prove the theorem for finitely generated FH -invariant subgroups of G , so we can assume that G is finitely generated and therefore nilpotent. Let \widehat{G} be the Mal'cev completion of G obtained by adjoining all roots of non-trivial elements of G (see [20] or, for example, [8, Section 10.1]). Forming the completion preserves the nilpotency class of a nilpotent subgroup. Every automorphism α of G can be canonically extended to an automorphism of \widehat{G} , which we denote by the same letter. The fixed-point subgroup $C_{\widehat{G}}(\alpha)$ is the completion of $C_G(\alpha)$. Applying this to FH , we see that as a group of automorphisms of \widehat{G} , it inherits the properties that $C_{\widehat{G}}(F) = 1$ and $C_{\widehat{G}}(H)$ is nilpotent of class c .

Under the Mal'cev correspondence, the radicable locally nilpotent torsion-free group \widehat{G} can be viewed as a locally nilpotent Lie algebra L (with the same underlying set) over the field of rational numbers \mathbb{Q} , with the Lie ring operations given by the inversions of the Baker–Campbell–Hausdorff formula. The automorphisms of \widehat{G} are automorphisms of L acting on the same set in the same way, and we denote them by the same letters. Thus, $C_L(F) = 0$ and $C_L(H)$ is nilpotent of class c . By Theorem 5.1 the Lie algebra L is nilpotent of (c, q) -bounded class. Hence the same is true for \widehat{G} and therefore also for G . \square

Lie rings

We now prove a similar theorem for arbitrary Lie rings. The additional conditions on the additive group of the Lie ring will be automatically satisfied when the theorem is later used in the proof of the main result on the nilpotency class of a finite group with a metacyclic Frobenius group of automorphisms. We also include the solvability result, which does not require those additional conditions.

Theorem 5.6. *Let FH be a Frobenius group with cyclic kernel F of order n and complement H of order q . Suppose that FH acts by automorphisms on a Lie ring L in such a way that $C_L(F) = 0$ and $C_L(H)$ is nilpotent of class c . Then*

- (a) *the Lie ring L is solvable of (c, q) -bounded derived length,*
- (b) *for some functions $u = u(c, q)$ and $v = v(c, q)$ depending only on c and q , the Lie subring $n^u L$ is nilpotent of class v , that is,*

$$\gamma_{v+1}(n^u L) = n^{u(v+1)}\gamma_{v+1}(L) = 0,$$

- (c) *moreover, L is nilpotent of (c, q) -bounded class in either of the following cases:*

- (i) *L is a Lie algebra over a field,*
- (ii) *the additive group of L is periodic (includes the case of L finite),*
- (iii) *n is invertible in the ground ring of L ,*
- (iv) *$nL = L$,*
- (v) *n is a prime-power.*

We included for completeness the case of algebras over a field, which is Theorem 5.1. It is not clear at the moment if the additional conditions (i)–(v) are really necessary. However, many important cases are covered, including those needed in our group-theoretic applications. The solvability result is included precisely because it does not require these conditions: of course, L is solvable of n -bounded derived length by Kreknin’s theorem, but we obtain a bound for the derived length in terms of $|H|$ and the nilpotency class of $C_L(H)$.

Proof. The proof is basically along the same lines as the proof of Theorem 5.1 for algebras; the complications arise because we no longer have a direct sum of eigenspaces forming a $(\mathbb{Z}/n\mathbb{Z})$ -grading.

We extend the ground ring by a primitive n th root of unity ω setting

$$\tilde{L} = L \otimes_{\mathbb{Z}} \mathbb{Z}[\omega].$$

The group FH acts in the natural way on \widetilde{L} and the action inherits the conditions that $C_{\widetilde{L}}(F) = 0$ and $C_{\widetilde{L}}(H)$ is nilpotent of class c . Since the conditions (i)–(v) would also hold for \widetilde{L} , and the conclusion of the theorem for L would follow from the same conclusion for \widetilde{L} , we can assume that $L = \widetilde{L}$ and the ground ring contains ω , a primitive n th root of 1.

Let $F = \langle \varphi \rangle$. For each $i = 0, \dots, n - 1$ we define the ‘eigenspace’ for ω^i as $L_i = \{x \in L \mid x^\varphi = \omega^i x\}$. Then

$$[L_i, L_j] \subseteq L_{i+j \pmod{n}} \quad \text{and} \quad nL \subseteq \sum_{i=0}^{n-1} L_i.$$

This is ‘almost a $(\mathbb{Z}/n\mathbb{Z})$ -grading’ – albeit of nL rather than of L , and although the sum is not necessarily direct, any linear dependence of elements from different L_i is annihilated by n :

$$\text{if } l_1 + l_2 + \dots + l_{n-1} = 0, \quad \text{then } nl_1 = nl_2 = \dots = nl_{n-1} = 0 \quad (5.3)$$

(see, for example, [7, Lemma 4.1.1]). We also have $L_0 = C_L(F) = 0$.

We shall mostly work with the FH -invariant subring $K = \sum_{i=0}^{n-1} L_i$.

As in the proof of Theorem 5.1, let $H = \langle h \rangle$ and

$$\varphi^{h^{-1}} = \varphi^r \quad \text{for some } 1 \leq r \leq n - 1,$$

so that n, q, r satisfy (4.1) and $L_i^h = L_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Using the same notation

$$u_{r^i k} = u_k^{h^i} \quad \text{and} \quad X_j = x_{a_j} + x_{ra_j} + \dots + x_{r^{q-1}a_j} \in C_L(H),$$

we have

$$[X_1, \dots, X_{c+1}] = 0.$$

Expand all the X_i and consider the resulting linear combination of commutators in the elements $x_{r^j a_i}$. Suppose that $[x_{a_1}, \dots, x_{a_{c+1}}] \neq 0$. In the ‘semisimple’ case, we find that the sum $K = \sum_{i=0}^{n-1} L_i$ is direct and there would have to be other terms in the same component $L_{a_1+\dots+a_{c+1}}$, which implies that

$$a_1 + \dots + a_{c+1} = r^{\alpha_1} a_1 + \dots + r^{\alpha_{c+1}} a_{c+1}$$

for some $\alpha_i \in \{0, 1, 2, \dots, q - 1\}$ not all of which are zeros, so that (a_1, \dots, a_{c+1}) is an r -dependent sequence. This would mean that $K = \sum_{i=0}^{n-1} L_i$ satisfies the selective c -nilpotency condition (4.2).

In the general case, where K may not be the direct sum of the L_i , it can happen that there are no other terms in the expanded expression with the same sum of

indices $a_1 + \dots + a_{c+1}$. But then the ‘almost linear independence’ (5.3) implies that $n[x_{a_1}, \dots, x_{a_{c+1}}] = 0$. Thus, in any case,

$$n[x_{d_1}, x_{d_2}, \dots, x_{d_{c+1}}] = 0 \quad \text{whenever } (d_1, \dots, d_{c+1}) \text{ is } r\text{-independent.} \quad (5.4)$$

We are now ready to prove part (a) on solvability. Let $y_{i_j} \in L_{i_j}$ be any elements of the ‘eigenspaces’ (under Index Convention). By Corollary 4.14 (a) the commutator $\delta_{f(c,q)}(y_{i_1}, y_{i_2}, \dots, y_{i_{2f(c,q)}})$ can be represented as a linear combination of commutators each of which contains either a subcommutator with zero modulo n sum of indices or a subcommutator $[g_{d_1}, g_{d_2}, \dots, g_{d_{c+1}}]$ with r -independent sequence (d_1, \dots, d_{c+1}) of indices. By hypothesis we have $L_0 = 0$, and $n[g_{d_1}, g_{d_2}, \dots, g_{d_{c+1}}] = 0$ by property (5.4). Hence,

$$n\delta_{f(c,q)}(y_{i_1}, y_{i_2}, \dots, y_{i_{2f(c,q)}}) = 0.$$

Thus, $nK^{(f(c,q))} = 0$ and therefore

$$n(nL)^{f(c,q)} = n^{2^{f(c,q)}+1}L^{(f(c,q))} = 0.$$

In particular, the additive group of $T = L^{(f(c,q))}$ is a periodic abelian group. We decompose it into the direct sum of Sylow subgroups

$$T = T_{p_1} \oplus T_{p_2} \oplus \dots \oplus T_{p_r},$$

where p_1, p_2, \dots, p_r are the prime divisors of n . The T_{p_k} are FH -invariant ideals and $[T_{p_i}, T_{p_j}] = 0$ for $i \neq j$.

Let $p \in \{p_1, p_2, \dots, p_r\}$ and $n = p^k s$, where $(p, s) = 1$. Let $\langle \psi \rangle$ be the Sylow p -subgroup of $F = \langle \varphi \rangle$ and let $\langle \varphi \rangle = \langle \psi \rangle \times \langle \chi \rangle$, where $s = |\chi|$. The fixed-point subring $C = C_{T_p}(\chi)$ is a ψ -invariant abelian p -group and $C_C(\psi) \subseteq C_L(\varphi) = 0$. The automorphism ψ of order p^t acting on an abelian p -group necessarily has non-trivial fixed points. Hence we get $C_{T_p}(\chi) = 0$. Thus the subring T_p admits the Frobenius group of automorphisms $\langle \chi \rangle H$ with cyclic kernel $\langle \chi \rangle$ and complement H of order q such that $C_{T_p}(\chi) = 0$ and $C_{T_p}(H)$ is nilpotent of class at most c . By the above argument,

$$0 = s(sT_p)^{(f(c,q))} = s^{2^{f(c,q)}+1}T_p^{(f(c,q))},$$

whence $T_p^{(f(c,q))} = 0$ for each prime p . Hence

$$T^{(f(c,q))} = (L^{(f(c,q))})^{(f(c,q))} = 0$$

and L is solvable of derived length at most $2f(c, q)$. The proof of part (a) is complete.

In the proof of the nilpotency statements, we need an analogue of Lemma 5.2.

Lemma 5.7. *There are (c, q) -bounded numbers l and m such that for every element $b \in \mathbb{Z}/n\mathbb{Z}$*

$$n^l [K, \underbrace{L_b, \dots, L_b}_m] \subseteq [[K, K], [K, K]].$$

Proof. We denote $[K, K] \cap L_a$ by $[K, K]_a$. Clearly, it suffices to show that

$$n^l [[K, K]_a, \underbrace{L_b, \dots, L_b}_{m-1}] \subseteq [[K, K], [K, K]]$$

for every $a, b \in \mathbb{Z}/n\mathbb{Z}$; we can of course assume that $a, b \neq 0$. First suppose that $(n, b) = 1$. If n is large enough, $n > N(c, q)$, then the order $o(b) = n$ is also large enough. Then by Corollary 4.14 (b), for the corresponding function $w = w(c, q)$, any commutator

$$[y_a, \underbrace{x_b, y_b, \dots, z_b}_w]$$

can be represented as a linear combination of commutators each of which has either a subcommutator with zero modulo n sum of indices or a subcommutator $[g_{u_1}, g_{u_2}, \dots, g_{u_{c+1}}]$ with r -independent sequence of indices (u_1, \dots, u_{c+1}) . Since $L_0 = 0$ and $n[g_{u_1}, g_{u_2}, \dots, g_{u_{c+1}}] = 0$ by property (5.4), we obtain that

$$n[[K, K]_a, \underbrace{L_b, \dots, L_b}_{m-1}] = 0,$$

so that we can put $m - 1 = w$. If $n \leq N(c, q)$, then there is a positive integer $k < n$ such that $a + kb = 0$ in $\mathbb{Z}/n\mathbb{Z}$. Then

$$[L_a, \underbrace{L_b, \dots, L_b}_k] \subseteq L_0 = 0,$$

so that we can put $m - 1 = N(c, q)$.

Now suppose that $(n, b) \neq 1$, and let s be a prime dividing both n and b . If s also divides a , then the result follows by induction on $|F|$ applied to the Lie ring $C_L(\varphi^{n/s}) = \sum_i L_{si}$ containing both L_a and L_b : this Lie subring is FH -invariant and φ acts on it as an automorphism of order n/s without non-trivial fixed points. The basis of this induction is the case of $n = |F|$ being a prime, where necessarily $(n, b) = 1$, which has already been dealt with above. The functions $l = l(c, q)$ and $m = m(c, q)$ remain the same in the induction step, so no dependence on F arises.

It remains to consider the case where the prime s divides both b and n and does not divide a . Using the same notation $x_{r^i k} = x_k^{h^i}$ (under Index Convention), we

have

$$\left[u_a + u_{ra} + \dots + u_{r^{q-1}a}, \underbrace{(v_b + v_{rb} + \dots + v_{r^{q-1}b}), \dots, (w_b + w_{rb} + \dots + w_{r^{q-1}b})}_c \right] = 0$$

for any c elements $v_b, \dots, w_b \in L_b$, because the sums are elements of $C_L(H)$.

By (4.1) any two indices $r^i a, r^j a$ here are different modulo s , while all the indices above the brace are divisible by s , which divides n . Hence by the ‘almost linear independence’ (5.3) we also have

$$n \left[u_a, \underbrace{(v_b + v_{rb} + \dots + v_{r^{q-1}b}), \dots, (w_b + w_{rb} + \dots + w_{r^{q-1}b})}_c \right] = 0. \tag{5.5}$$

Now, let Z denote the additive subgroup of $\sum_{i=0}^{q-1} L_{r^i b}$ generated by all the sums $x_b + x_{rb} + \dots + x_{r^{q-1}b}$ over $x_b \in L_b$ (in general Z may not contain all the fixed points of H on $\sum_{i=0}^{q-1} L_{r^i b}$). Then (5.5) means that

$$n \left[L_a, \underbrace{Z, \dots, Z}_c \right] = 0.$$

Applying φ^j we also obtain

$$n \left[L_a, \underbrace{Z^{\varphi^j}, \dots, Z^{\varphi^j}}_c \right] = 0. \tag{5.6}$$

We now apply Lemma 5.3 with $M = L_b + L_{rb} + \dots + L_{r^{q-1}b}$ and $m = q$ to $w = v_b + v_{rb} + \dots + v_{r^{q-1}b} \in Z$ for any $v_b \in L_b$. As a result we obtain that $n^{l_0} L_b \subseteq \sum_{j=0}^{q-1} Z^{\varphi^j}$ for some (c, q) -bounded number l_0 .

We now claim that

$$\begin{aligned} n^{l_0(q(c-1)+1)+1} [[K, K]_a, \underbrace{L_b, \dots, L_b}_{q(c-1)+1}] &= n [[K, K]_a, \underbrace{n^{l_0} L_b, \dots, n^{l_0} L_b}_{q(c-1)+1}] \\ &\subseteq [[K, K], [K, K]]. \end{aligned}$$

Indeed, after replacing $n^{l_0} L_b$ with $\sum_{j=0}^{q-1} Z^{\varphi^j}$ and expanding the sums, in each commutator of the resulting linear combination we can freely permute modulo $[[K, K], [K, K]]$ the entries Z^{φ^j} . Since there are sufficiently many of them, we can place at least c of the same $Z^{\varphi^{j_0}}$ for some j_0 right after $[L, L]_a$ at the beginning, which gives 0 by (5.6). □

We now prove part (b) of the theorem. Since L is solvable of (c, q) -bounded derived length by (a), we can use induction on the derived length of L . If L is abelian, there is nothing to prove. Assume that L is metabelian, which is the main part of the proof. Consider $K = \sum L_i$. Let $l = l(c, q)$ and $m = m(c, q)$ be as in Lemma 5.7; put $g = (m - 1)(q^{c+1} + c) + 2$. For any sequence of g non-zero elements (a_1, \dots, a_g) in $\mathbb{Z}/n\mathbb{Z}$ consider the commutator $[[K, K]_{a_1}, L_{a_2}, \dots, L_{a_g}]$. If the sequence (a_1, \dots, a_g) contains an r -independent subsequence of length $c + 1$ that starts with a_1 , by permuting the L_{a_i} we can assume that a_1, \dots, a_{c+1} is the r -independent subsequence. Then $n[[K, K]_{a_1}, L_{a_2}, \dots, L_{a_g}] = 0$ by (5.4). If the sequence (a_1, \dots, a_g) does not contain an r -independent subsequence of length $c + 1$ starting with a_1 , then by Lemma 4.5 the sequence (a_1, \dots, a_g) contains at most $q^{c+1} + c$ different values. The number g was chosen big enough to guarantee that either the value of a_1 occurs in (a_1, \dots, a_g) at least $m + 1$ times or, else, another value, different from a_1 , occurs at least m times. Using the fact that $[x, y, z] = [x, z, y]$ for $x \in [K, K]$, we can assume that $a_2 = \dots = a_{m+1}$, in which case it follows from Lemma 5.7 that $n^l[[K, K]_{a_1}, L_{a_2}, \dots, L_{a_g}] = 0$. Thus, in any case $n^l \gamma_{g+1}(K) = 0$. Clearly, we can also choose a (c, q) -bounded number l_1 such that $\gamma_{g+1}(n^{l_1} K) = 0$. As $nL \subseteq K$, we get $\gamma_{g+1}(n^{l_1+1} L) = 0$, as required.

Now suppose that the derived length of L is at least 3. By induction hypothesis, $\gamma_{g'+1}(n^{l_2}[L, L]) = 0$ for some (c, q) -bounded numbers g' and l_2 . Since $g \geq 4$, we can choose a (c, q) -bounded number y such that $y(g + 1 - 4) \geq (g + 1)(l_1 + 1) \Leftrightarrow y(g + 1) \geq (l_1 + 1)(g + 1) + 4y$. We set $x = \max\{y, l_2/2\}$. Then for $M = n^x L$ we have both

$$\begin{aligned} \gamma_{g+1}(M) &= \gamma_{g+1}(n^x L) = n^{x(g+1)} \gamma_{g+1}(L) \\ &\subseteq n^{(l_1+1)(g+1)+4x} \gamma_{g+1}(L) \subseteq n^{4x} [[L, L], [L, L]] \\ &= [[M, M], [M, M]], \end{aligned}$$

by the metabelian case above, and

$$\gamma_{g'+1}([M, M]) = \gamma_{g'+1}(n^{2x}[L, L]) \subseteq \gamma_{g'+1}(n^{l_2}[L, L]) = 0.$$

Together, these give the nilpotency of $M = n^x L$ of (c, q) -bounded class by the Lie ring analogue (3.1) of P. Hall's theorem.

We now proceed with nilpotency of L itself in cases (i)–(v) of part (3).

Case (i). The case of a Lie algebra was already settled in Theorem 5.1.

Cases (iii) and (iv). In these two cases we have $L = nL = n^i L$ for any i . By part (b),

$$\gamma_{v+1}(L) = \gamma_{v+1}(n^u L) = 0.$$

Case (v). Let $n = p^k$ for a prime p . Then

$$\gamma_{v+1}((p^k)^u L) = p^{ku(v+1)} \gamma_{v+1}(L) = 0$$

by part (b), so that the additive group $\gamma_{v+1}(L)$ is a p -group. Since in this case F is a cyclic p -group acting fixed-point-freely, we must have $\gamma_{v+1}(L) = 0$.

Case (ii). Since here the additive group of L is periodic, it decomposes into the direct sum of Sylow subgroups $L = \bigoplus_p T_p$, which are FH -invariant ideals satisfying $[T_q, T_p] = 0$ for different prime numbers p, q . It is sufficient to prove that every ideal T_p is nilpotent of (c, q) -bounded class. So we can assume that the additive group of L is a p -group. If p does not divide n , then $nL = L$ and the assertion follows from (iv). If p divides n , then the Hall p' -subgroup $\langle \chi \rangle$ of F acts fixed-point-freely on L , so that L admits the Frobenius group of automorphisms $\langle \chi \rangle H$ with $C_L(\chi) = 0$. Since $|\chi|L = L$, it remains to apply (iv) to $\langle \chi \rangle H$. \square

Finite groups

As a consequence we obtain our main result on the bound for the nilpotency class of a finite group with a metacyclic Frobenius group of automorphisms.

Theorem 5.8. *Suppose that a finite group G admits a Frobenius group of automorphisms FH with cyclic kernel F of order n and complement H of order q such that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class c . Then G is nilpotent of (c, q) -bounded class.*

The assumption of the kernel F being cyclic is essential: see the examples at the end of the section.

Proof. The group G is nilpotent by Theorem 2.7 (c). We have to bound the nilpotency class of G in terms of c and q . Consider the associated Lie ring of G

$$L(G) = \bigoplus_{i=1}^m \gamma_i / \gamma_{i+1},$$

where m is the nilpotency class of G and the γ_i are terms of the lower central series of G . The nilpotency class of G coincides with that of $L(G)$. The action of the group FH on G naturally induces an action of FH on $L(G)$. By the definition of the associated Lie ring,

$$C_{L(G)}(H) = \bigoplus_i C_{\gamma_i / \gamma_{i+1}}(H),$$

which by Theorem 2.3 is equal to

$$\bigoplus_i C_{\gamma_i}(H)\gamma_{i+1}/\gamma_{i+1}.$$

The Lie ring products in $L(G)$ are defined for elements of the γ_i/γ_{i+1} in terms of the images of the group commutators and then extended by linearity. If c is the nilpotency class of $C_G(H)$, then any group commutator of weight $c+1$ in elements of $C_G(H)$ is trivial; hence any Lie ring commutator of weight $c+1$ in elements of $C_{\gamma_i}(H)\gamma_{i+1}/\gamma_{i+1}$ is also trivial. Since these elements generate $C_{L(G)}(H)$, this subring is nilpotent of class at most c . Because F acts fixed-point-freely on every quotient γ_i/γ_{i+1} by Lemma 2.2, it follows that $C_{L(G)}(F) = 0$.

We now deduce from Theorem 5.6 (c) that $L(G)$ is nilpotent of (c, q) -bounded class. Since the nilpotency class of our group G is equal to that of $L(G)$, the result follows. \square

Examples

Example 5.9. The simple 3-dimensional Lie algebra L of characteristic $\neq 2$ with basis e_1, e_2, e_3 and structure constants $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$ admits the Frobenius group of automorphisms FH with F non-cyclic of order 4 and H of order 3 such that $C_L(F) = 0$ and $C_L(H)$ is one-dimensional. Namely, $F = \{1, f_1, f_2, f_3\}$, where $f_i(e_i) = e_i$ and $f_i(e_j) = -e_j$ for $i \neq j$, and $H = \langle h \rangle$ with $h(e_i) = e_{i+1 \pmod{3}}$.

Example 5.10. One can also modify the above example to produce nilpotent Lie rings L of unbounded derived length admitting the same non-metacyclic Frobenius group of automorphisms FH of order 12 such that $C_L(F) = 0$ and $C_L(H)$ is abelian ('one-dimensional'). Namely, let the additive group of L be the direct sum of three copies of $\mathbb{Z}/p^m\mathbb{Z}$ for some prime $p \neq 2$ with generators e_1, e_2, e_3 and let the structure constants of L be $[e_1, e_2] = pe_3, [e_2, e_3] = pe_1, [e_3, e_1] = pe_2$. The action of FH is 'the same', that is, $F = \{1, f_1, f_2, f_3\}$, where $f_i(e_i) = e_i$ and $f_i(e_j) = -e_j$ for $i \neq j$, and $H = \langle h \rangle$ with $h(e_i) = e_{i+1 \pmod{3}}$. Then we have $C_L(F) = 0$ and $C_L(H) = \langle e_1 + e_2 + e_3 \rangle$. It is easy to see that L is nilpotent of class exactly m , while the derived length is approximately $\log m$.

Example 5.11. If in the preceding example we choose $p > m$, then the Lazard correspondence based on the 'truncated' Baker–Campbell–Hausdorff formula (cf. [15]; also see, for example, [8, Section 10.2]) transforms L into a finite p -group P of the same derived length as L , which admits the same group of automorphisms FH with $C_P(F) = 1$ and $C_P(H)$ cyclic.

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