Introduction to Linear Dynamical Systems and Linear Control Strategies

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Outline

- Linear Time Invariant Systems
- Linear Time Invariant Feedback Controls
  - Pole Placement Approach
  - State Feedback
  - Output Feedback
- Linear Multirate Systems
- Control of Linear Multirate Systems via Filter Banks Approach
- Conclusions
- Questions and Answers
Linear Time Invariant Systems

- Definition of linear systems
  \[ \sum_{i=0}^{N} a_i y_i(k) = T \left( \sum_{i=0}^{N} a_i u_i(k) \right) \]

- Definition of time invariant systems
  \[ y(k-1) = T(u(k-1)) \]

- Definition of linear time invariant systems
  
  A system is both linear and time invariant.
Linear Time Invariant Systems

- Definition of an impulse response
  \[ h(k) \equiv T(\delta(k)) \]
  where \[ \delta(k) \equiv \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases} \]

- Definition of a frequency response
  \[ y(k) \equiv T(e^{j\omega k}) \]
Linear Time Invariant Systems

- Properties of linear time invariant systems
  - A system is linear and time invariant if and only if
    \[ y(n) = \sum_{k \in \mathbb{Z}} h(k)u(n - k) \]
  - A system is linear and time invariant if and only if
    \[ Y(z) = H(z)U(z) \]
    where
    \[ Y(z) = \sum_{n \in \mathbb{Z}} y(n)z^{-n} \]
    \[ H(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n} \]
    \[ U(z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n} \]
Linear Time Invariant Systems

- Characterization of linear time invariant systems
  - Constant linear coefficients difference equations
    \[ \sum_{i=0}^{N} a_i y(k-i) = \sum_{j=0}^{M} b_j u(k-j) \]
  - Transfer function
    \[ H(z) = \frac{\sum_{j=0}^{M} b_j z^{-j}}{\sum_{i=0}^{N} a_i z^{-i}} \]
  - State space representation
    \[
    \begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \\
    y(k) &= Cx(k) + Du(k)
    \end{align*}
    \]
Linear Time Invariant Systems

- Responses

\[ x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) \quad \forall k \geq 1 \]
\[ y(k) = CA^k x(0) + C \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) + Du(k) \quad \forall k \geq 1 \]

zero input response  zero state response

\[ y(n) = \sum_{\forall k \in \mathbb{Z}} h(k) u(n-k) \]
Linear Time Invariant Systems

- **Similarity transforms**
  - **Define**
    - $\tilde{x}(k) \equiv T^{-1}x(k)$
    - $\tilde{A} \equiv T^{-1}AT$
    - $\tilde{B} \equiv T^{-1}B$
    - $\tilde{C} \equiv CT$
  - **then**
    - $\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}u(k)$
    - $y(k) = \tilde{C}\tilde{x}(k) + Du(k)$
Linear Time Invariant Systems

- Only three types of behaviors for autonomous response:
  - converge to zero (all system poles are strictly inside the unit circle.)
  - oscillates (Some system poles are on the unit circle, while all other system poles are strictly inside the unit circle.)
  - diverge to infinity (Some system poles are outside the unit circle.)
Linear Time Invariant Systems

- Autonomous responses
Linear Time Invariant Systems

Effects on initial conditions

Behaviors only depend on the system poles, not on initial conditions.
Pole placement

Plant transfer function

\[ H(z) = \frac{N_H(z)}{D_H(z)} \]

Controller transfer function

\[ F(z) = \frac{N_F(z)}{D_F(z)} \]

\[ T(z) = \frac{H(z)}{1 + H(z)F(z)} = \frac{N_H(z)D_F(z)}{N_H(z)N_F(z) + D_H(z)D_F(z)} \]

\[ N_H(z)N_F(z) + D_H(z)D_F(z) \text{ is stable.} \]
Linear Time Invariant Feedback Controls

- **State feedback**
  - Plant state space matrices \((A, B, C, D)\)
  - Controller state space matrices \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\)

\[
x(k+1) = Ax(k) + B(u(k) - \tilde{y}(k)) \\
y(k) = Cx(k) + D(u(k) - \tilde{y}(k)) \\
\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}x(k) \\
\tilde{y}(k) = \tilde{C}\tilde{x}(k) + \tilde{D}x(k) \\
y(k) = Cx(k) + D\left(u(k) - \left(\tilde{C}\tilde{x}(k) + \tilde{D}x(k)\right)\right) \\
= \left(C - DD\right)x(k) - D\tilde{C}\tilde{x}(k) + Du(k) \\
x(k+1) = Ax(k) + B\left(u(k) - \left(\tilde{C}\tilde{x}(k) + \tilde{D}x(k)\right)\right) \\
= \left(A - BD\right)x(k) - B\tilde{C}\tilde{x}(k) + Bu(k)
\]
Linear Time Invariant Feedback Controls

- **State feedback**

\[
\begin{bmatrix}
\dot{x}(k+1) \\
\ddot{x}(k+1)
\end{bmatrix} =
\begin{bmatrix}
A - B\tilde{D} & -B\tilde{C} \\
\tilde{B} & \tilde{A}
\end{bmatrix}
\begin{bmatrix}
x(k) \\
\ddot{x}(k)
\end{bmatrix} +
\begin{bmatrix}
B \\
0
\end{bmatrix} u(k)
\]

\[
y(k) =
\begin{bmatrix}
C - D\tilde{D} & -D\tilde{C}
\end{bmatrix}
\begin{bmatrix}
x(k) \\
\ddot{x}(k)
\end{bmatrix} + Du(k)
\]

\[
\begin{bmatrix}
A - B\tilde{D} & -B\tilde{C} \\
\tilde{B} & \tilde{A}
\end{bmatrix}
\]

is stable.
Linear Time Invariant Feedback Controls

Output feedback

- Plant state space matrices \((A, B, C, D)\)
- Controller state space matrices \((\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\)

\[
\begin{align*}
x(k+1) &= Ax(k) + B(u(k) - \tilde{y}(k)) \\
y(k) &= Cx(k) + Du(k) - \tilde{y}(k) \\
\tilde{x}(k+1) &= \tilde{A}\tilde{x}(k) + \tilde{B}y(k) \\
\tilde{y}(k) &= \tilde{C}\tilde{x}(k) + \tilde{D}y(k) \\
y(k) &= Cx(k) + D(u(k) - (\tilde{C}\tilde{x}(k) + \tilde{D}y(k))) \\
&= Cx(k) - D\tilde{C}\tilde{x}(k) + Du(k) - D\tilde{D}y(k) \\
y(k) &= \left(I + D\tilde{D}\right)^{-1} \left(Cx(k) - D\tilde{C}\tilde{x}(k) + Du(k)\right) \\
&= \left(I + D\tilde{D}\right)^{-1} Cx(k) - \left(I + D\tilde{D}\right)^{-1} D\tilde{C}\tilde{x}(k) + \left(I + D\tilde{D}\right)^{-1} Du(k)
\end{align*}
\]
Linear Time Invariant Feedback Controls

Output feedback

\[ \tilde{y}(k) = \tilde{C}\tilde{x}(k) + \tilde{D}\left(\left(I + \tilde{D}\tilde{D}\right)^{-1} Cx(k) - \left(I + \tilde{D}\tilde{D}\right)^{-1} \tilde{D}\tilde{C}\tilde{x}(k) + \left(I + \tilde{D}\tilde{D}\right)^{-1} Du(k)\right) \]

\[ = \tilde{D}\left(I + \tilde{D}\tilde{D}\right)^{-1} Cx(k) + \left(I - \tilde{D}\left(I + \tilde{D}\tilde{D}\right)^{-1} D\right)\tilde{C}\tilde{x}(k) + \tilde{D}\left(I + \tilde{D}\tilde{D}\right)^{-1} Du(k) \]

\[ x(k+1) = Ax(k) + Bu(k) - \left(\tilde{D}\left(I + \tilde{D}\tilde{D}\right)^{-1} Cx(k) + \left(I - \tilde{D}\left(I + \tilde{D}\tilde{D}\right)^{-1} D\right)\tilde{C}\tilde{x}(k) + \tilde{D}\left(I + \tilde{D}\tilde{D}\right)^{-1} Du(k)\right) \]

\[ = \left(A - \tilde{B}\left(I + \tilde{D}\tilde{D}\right)^{-1} C\right)x(k) - \left(B - \tilde{B}\left(I + \tilde{D}\tilde{D}\right)^{-1} D\right)u(k) \]

\[ \tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}\left(I + \tilde{D}\tilde{D}\right)^{-1} Cx(k) - \left(I + \tilde{D}\tilde{D}\right)^{-1} \tilde{D}\tilde{C}\tilde{x}(k) + \left(I + \tilde{D}\tilde{D}\right)^{-1} Du(k) \]

\[ = \tilde{B}\left(I + \tilde{D}\tilde{D}\right)^{-1} Cx(k) + \left(\tilde{A} - \tilde{B}\left(I + \tilde{D}\tilde{D}\right)^{-1} D\tilde{C}\right)\tilde{x}(k) + \tilde{B}\left(I + \tilde{D}\tilde{D}\right)^{-1} Du(k) \]
Linear Time Invariant Feedback Controls

Output feedback

\[
\begin{bmatrix}
    x(k+1) \\
    \tilde{x}(k+1)
\end{bmatrix} = \begin{bmatrix}
    A - B\tilde{D}(I + D\tilde{D})^{-1}C & -B\left(I - \tilde{D}(I + D\tilde{D})^{-1}D\right)\tilde{C} \\
    \tilde{B}(I + D\tilde{D})^{-1}C & \left(\tilde{A} - \tilde{B}(I + D\tilde{D})^{-1}D\tilde{C}\right)
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    \tilde{x}(k)
\end{bmatrix} + \begin{bmatrix}
    B - B\tilde{D}(I + D\tilde{D})^{-1}D \\
    \tilde{B}(I + D\tilde{D})^{-1}D
\end{bmatrix} u(k)
\]

\[
\tilde{y}(k) = \begin{bmatrix}
    \tilde{D}(I + D\tilde{D})^{-1}C & \left(I - \tilde{D}(I + D\tilde{D})^{-1}D\right)\tilde{C}
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    \tilde{x}(k)
\end{bmatrix} + \tilde{D}(I + D\tilde{D})^{-1}Du(k)
\]

\[
\begin{bmatrix}
    A - B\tilde{D}(I + D\tilde{D})^{-1}C & -B\left(I - \tilde{D}(I + D\tilde{D})^{-1}D\right)\tilde{C} \\
    \tilde{B}(I + D\tilde{D})^{-1}C & \left(\tilde{A} - \tilde{B}(I + D\tilde{D})^{-1}D\tilde{C}\right)
\end{bmatrix}
\]

is stable.
Linear Multirate Systems

Definition

\[ y(k) = \sum_{l \in \mathbb{Z}} g(k, l)u(l) \quad \forall k \in \mathbb{Z} \]

where

\[ g(k, l) = g(k + m, l + n) \quad \forall k, l \in \mathbb{Z} \]

- Input shifts by \( n \) samples, output shifts by \( m \) samples.
Linear Multirate Systems

- **Examples:**
  - Rate changers/sampled data systems
    
    ![rate changer diagram]

  - Filter banks
    
    ![filter bank diagram]
**Linear Multirate Systems**

- **Realization**
  - A linear multirate system can be realized by a filter bank system.
  - Define a blocked input signal as
    \[ u(k) \equiv [u(nk) \quad \cdots \quad u(nk + n - 1)]^T \]
  - Define a block output signal as
    \[ y(k) \equiv [y(mk) \quad \cdots \quad y(mk + m - 1)]^T \]
  - Input shifts by \( n \) samples, the blocked input signal shifts by 1 sample. Output shifts by \( m \) samples, the blocked output signal shifts by 1 sample.
  - Hence, there exists an \( m \times n \) transfer matrix \( H(z) \) such that
    \[ Y(z) = H(z)X(z) \]
Linear Multirate Systems

- **Realization**

\[ y[n] = \sum_{i=0}^{n-1} h_i[n] u[n-i] \]

\[ H(z) = \sum_{i=0}^{n-1} h_i[n] z^{-i} \]

\[ u[n] \rightarrow \downarrow n \rightarrow h_0[n] \rightarrow \downarrow n \rightarrow \cdots \rightarrow \downarrow n \rightarrow h_{n-1}[n] \rightarrow \uparrow m \rightarrow y[n] \]

\[ u[n] \rightarrow \downarrow n \rightarrow \cdots \rightarrow \downarrow n \rightarrow y[n] \]
Linear Multirate Systems

Realization

- Denote \( f[kn - lm] = g[k, l] \quad \forall k, l \in \mathbb{Z} \)
- Define the map \( I : \{0,1,\ldots,m-1\} \times \mathbb{Z} \to \mathbb{Z} \) such that \( I(k,l) = kn - lm \)
- \( I \) is bijective if and only if \( m \) and \( n \) is co-prime. Or in other words, \( I \) is bijective if and only if the highest common factor of \( m \) and \( n \) is 1.
Linear Multirate Systems

- Realization

Figure 1a. Mapping from $g[n,k]$ to $f[k]$ when $m$ and $n$ are co-prime.
Linear Multirate Systems

- Realization

Figure 1b. Mapping from $g[n,k]$ to $f[k]$ when $n=cm$. 
Linear Multirate Systems

- Realization

Figure 1c. Mapping from $g[n,k]$ to $f[k]$ when $m=cn$. 
Linear Multirate Systems

- Realization

Figure 1d. Mapping from $g[n,k]$ to $f[k]$ when $m=n$. 
A linear multirate system is equivalent to a rate changer if and only if $m$ and $n$ is co-prime. That is:
Linear Multirate Systems

Properties

- A linear multirate system is stable if and only if $h_i[n]$ for $i=0,1,\ldots,n-1$ are all stable.
- A linear multirate system is finite impulse response if and only if $h_i[n]$ for $i=0,1,\ldots,n-1$ are all finite impulse response.
Linear Multirate Systems

Realization

Block decimators (decimation ratio $M$ and block length $L$)

$$y[Lk + j] = x[kML + j] \text{ for } j=0,1,\ldots,L-1 \text{ and } k \in \mathbb{Z}.$$
Linear Multirate Systems

Realization

Block expanders (expansion ratio $M$ and block length $L$)

$$y[k] = \begin{cases} 
  x \left( \frac{k - \text{mod}(k, ML)}{M} + \text{mod}(k, ML) \right) & \text{if } k - \text{mod}(k, ML) \leq k < k - \text{mod}(k, ML) + L \\
  0 & \text{if } k - \text{mod}(k, ML) + L \leq k < k + ML - \text{mod}(k, ML) 
\end{cases}$$
Realization

∀m,n ∈ ℤ⁺ (no matter m and n are co-prime or not), all linear multirate systems (shifting input by n samples resulting to shifting an output by m samples) can be represented via a series cascade of ↑m, followed by an LTI filter with an impulse response f[k], and then followed by ↓(n,m).
Linear Multirate Systems

- **Realization**

- The input output relationship of all linear multirate systems is
  \[ y[km+i] = \sum_{l=-\infty}^{+\infty} g[i, l-\text{kn}] u[l], \quad \forall k,l \in \mathbb{Z}, \forall m,n \in \mathbb{Z}^+, \text{ and } i=0,1,\ldots,m-1. \]

- The input output relationship of the system with block sampler is
  \[ y[km+i] = \sum_{l=-\infty}^{+\infty} f[kmn-ml+i] u[l], \quad \forall k,l \in \mathbb{Z}, \forall m,n \in \mathbb{Z}^+, \text{ and } i=0,1,\ldots,m-1. \]

- \( \forall k,l \in \mathbb{Z}, \forall m,n \in \mathbb{Z}^+, \text{ and } i=0,1,\ldots,m-1, \) the mapping from \( \{0,1,\ldots,m-1\} \times \mathbb{Z} \) to \( \mathbb{Z} \), where \( [i,l-\text{kn}] \in \{0,1,\ldots,m-1\} \times \mathbb{Z} \) and \( knm-ml+i \in \mathbb{Z} \) is bijective.
Realization

Hence, \( \forall k, l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+ \) and \( i=0, 1, \ldots, m-1 \), there exists a unique time index \( kmn-ml+i \) corresponding to the time index \( [i, l-kn] \).

As a result, there exists an LTI filter with an impulse response \( f[k] \) satisfying \( f[kmn-ml+i]=g[i, l-kn] \), \( \forall k, l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+ \) and \( i=0, 1, \ldots, m-1 \), that the linear multirate rate systems and the system with block sampler are input output equivalent.
Linear Multirate Systems

- **Realization**
  - $\forall m, n \in \mathbb{Z}^+$ (no matter $m$ and $n$ are co-prime or not), all linear multirate rate systems (with shifting input by $n$ samples resulting to shifting an output by $m$ samples) can be represented via a series cascade of $\uparrow(m, n)$, followed by an LTI filter with an impulse response $f[k]$, and then followed by $\downarrow n$. 
Realization

The input output relationship of all linear multirate systems is\[ y[k] = \sum_{l=-\infty}^{\infty} \sum_{i=0}^{n-1} g[k, nl + i] u[nl + i], \forall k, l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+ \text{ and } i=0, 1, \ldots, n-1.\]

The input output relationship of the system with block sampler is\[ y[k] = \sum_{l=-\infty}^{\infty} \sum_{i=0}^{n-1} f[kn - mnl - i] u[nl + i], \forall k, l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+ \text{ and } i=0, 1, \ldots, n-1.\]

\(\forall l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+, k \in \{0, 1, \ldots, m-1\} \text{ and } i \in \{0, 1, \ldots, n-1\},\) the mapping from \(\{0, 1, \ldots, m-1\} \times \mathbb{Z}\) to \(\mathbb{Z}\), where \([k, nl+i] \in \{0, 1, \ldots, m-1\} \times \mathbb{Z}\) and \(kn-mnl-i \in \mathbb{Z}\) is bijective.
Linear Multirate Systems

- Realization
  - Hence, \( \forall l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+, k \in \{0, 1, \ldots, m-1\} \) and \( i \in \{0, 1, \ldots, n-1\} \), there exists a unique time index \( kn-mnl-i \) corresponding to the time index \([k, nl+i]\).
  - As a result, there exists an LTI filter with an impulse response \( f[k] \) satisfying \( f[kn-mnl-i]=g[k, nl+i] \), \( \forall k, l \in \mathbb{Z}, \forall m, n \in \mathbb{Z}^+ \) and \( i=0, 1, \ldots, n-1 \), that the linear multirate rate systems and the system with block sampler are input output equivalent.
Linear Multirate Systems

- **Properties**

- A linear multirate system is stable if and only if \( f[n] \) is stable.
- A linear multirate system is finite impulse response if and only if \( f[n] \) is finite impulse response.
Control of Linear Multirate Systems via Filter Banks Approach

- **Plant model**

- **Controller model**
Control of Linear Multirate Systems via Filter Banks Approach

- Closed loop system model

\[ U(z) \equiv \sum_{k} u(k)z^{-k} \]
\[ U_i(z) \equiv \sum_{k} u(nk+i)z^{-k} \]
\[ U(z) = \sum_{i=0}^{n-1} z^{-i}U_i(z^n) \]
\[ P(z) \equiv \sum_{k} p(k)z^{-k} \]
\[ P_i(z) \equiv \sum_{k} p(nk+i)z^{-k} \]
\[ P(z) = \sum_{i=0}^{n-1} z^{-i}P_i(z^n) \]

\[ U_p(z) \equiv [U_0(z) \ldots U_{n-1}(z)]^T \]
\[ P_p(z) \equiv [P_0(z) \ldots P_{n-1}(z)]^T \]
Closed loop system model

\[ Y(z) \equiv \sum_{k} y(k)z^{-k} \]

\[ Y_i(z) \equiv \sum_{k} y(mk + i)z^{-k} \]

\[ Y(z) = \sum_{i=0}^{m-1} z^{-i} Y_i(z^m) \]

\[ Y_p(z) \equiv [Y_0(z) \quad \cdots \quad Y_{m-1}(z)]^T \]

\[ G(z)H(z)(U_p(z) - P_p(z)) = P_p(z) \]

\[ P_p(z) = (I + G(z)H(z))^{-1} G(z)H(z)U_p(z) \]
Control of Linear Multirate Systems via Filter Banks Approach

- Closed loop system model

\[ Y_p(z) = H(z)(U_p(z) - P_p(z)) \]

\[ = H(z)(I - (I + G(z)H(z))^{-1}G(z)H(z))U_p(z) \]

\[ H(z)(I - (I + G(z)H(z))^{-1}G(z)H(z)) \text{ is stable.} \]
Conclusions

- Only three types of behaviors for autonomous response of linear time invariant systems.
- Behaviors of linear time invariant systems only depend on the system poles, not on initial conditions.
- Stability conditions based on pole placement, state feedback and output feedback of linear time invariant systems are derived.
- Linear multirate systems can be realized via a filter bank.
- When the input rate and the output rate is co-prime, then linear multirate systems can be realized via linear rate changers. Otherwise, they can be realized via block samplers.
- Stability conditions for linear multirate feedback systems are derived based on filter bank approach.
Questions and Answers

Thank you!
Let me think…