GROUPS OF $p$-DEFICIENCY ONE

ANITHA THILLAISUNDARAM

Abstract. In a previous paper, Button and I proved that all finitely presented groups of $p$-deficiency greater than one are $p$-large. Here I prove that groups with a finite presentation of $p$-deficiency one possess a finite index subgroup that surjects onto the integers. This implies that these groups do not have Kazhdan's property (T). Additionally, I prove that the aforementioned result of Button and myself implies a result of Lackenby.

Introduction

This paper continues from my joint paper with Button [5]. Throughout this paper, $p$ denotes a prime. We recall the following.

Definition. [12, 13] Let $G$ be a finitely generated group. Say $G \cong \langle X \mid R \rangle$, with $|X|$ finite. For a prime $p$, the $p$-deficiency of $G$ with presentation $\langle X \mid R \rangle$ is

$$\text{def}_p(G; X, R) = |X| - \sum_{r \in R} p^{-\nu_p(r)},$$

where $\nu_p(r) = \max \left\{ k \geq 0 \mid \exists w \in F(X), w^p^k = r \right\}$. The $p$-deficiency of $G$ is then defined to be the supremum of $\text{def}_p(G; X, R)$ over all presentations $\langle X \mid R \rangle$ of $G$ with $|X|$ finite.

This is similar to an older concept:

Definition. The deficiency of a group $G$ is

$$\text{def}(G) = \sup_{\langle X \mid R \rangle} \{|X| - |R| : G \cong \langle X \mid R \rangle\}.$$

Recent and interesting developments in $p$-deficiency (and in particular, on groups of $p$-deficiency one) were made by Barnea & Schlage-Puchta [2]. The reader should note our different definition of $p$-deficiency, which is larger by 1 compared to Schlage-Puchta’s definition [13].

We recall the concepts of largeness and $p$-largeness.

Definition. [10] Let $G$ be a group, and let $p$ be a prime. Then

- $G$ is large if some (not necessarily normal) subgroup with finite index admits a non-abelian free quotient;
- $G$ is $p$-large if some normal subgroup with index a power of $p$ admits a non-abelian free quotient.

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The main result of [5] is the following.

**Theorem 1.** ([5], Theorem 2.2) Let \( p \) be a prime. If \( G \) is a finitely presented group with \( p \)-deficiency greater than one, then \( G \) is \( p \)-large.

Theorem 1 is proved using results of Lackenby ([10], Theorem 1.15) and Schlage-Puchta [12, 13]. See [5] for details of the proof.

By Corollary 2.1 of [5], groups with a finite presentation of \( p \)-deficiency one are infinite. In this paper, we prove the following.

**Theorem.** Let \( \Gamma \) be a finitely presented group with a presentation of \( p \)-deficiency equal to one, for some prime \( p \). Then \( \Gamma \) has a finite index subgroup \( H \) that surjects onto \( \mathbb{Z} \).

The Related Burnside Problem ([8], Problem 8.52) asks whether or not there exist infinite finitely presented torsion groups. The theorem above extends Corollary 2.4 of [5] to give the following response to the Related Burnside Problem.

**Corollary.** Let \( G \) be an infinite finitely presented group with a presentation of \( p \)-deficiency greater than or equal to one, for some prime \( p \). Then \( G \) is not torsion.

We note here the definition of Kazhdan’s property (T). See [3] for more information on property (T).

**Definition.** [3] Let \( \Gamma \) be a finitely generated group.

(a) Given a unitary representation \( V \) of \( \Gamma \) and a generating set \( S \) of \( \Gamma \), we define \( \kappa(\Gamma; S; V) \) to be the largest \( \varepsilon \geq 0 \) such that for any \( v \in V \) there exists \( s \in S \) with \( ||sv - v|| \geq \varepsilon||v|| \).

(b) Given a generating set \( S \) of \( \Gamma \), the Kazhdan constant \( \kappa(\Gamma; S) \) is defined to be the infimum of the set \( \{\kappa(\Gamma; S; V)\} \) where \( V \) runs over all unitary representations of \( \Gamma \) without non-zero invariant vectors.

(c) The group \( \Gamma \) is called a Kazhdan group (equivalently \( \Gamma \) is said to have Kazhdan's property (T)) if \( \kappa(\Gamma; S) > 0 \) for some (hence any) finite generating set \( S \) of \( \Gamma \).

It is proved in [3] (Corollaries 1.3.6 & 1.7.2) that if a group has property (T), then its finite index subgroups must have finite abelianization.

Lastly, we include the definition of linear growth of mod \( p \) homology for later use.

**Definition.** [10] We say that a collection of finite index subgroups \( \{G_i\} \) has linear growth of mod \( p \) homology if

\[
\inf_i \frac{d_p(G_i)}{[G : G_i]} > 0.
\]

Section 1 of this paper presents the proof of our main result. The corollaries of our main result are the content of Section 2. We finish with Section 3 which gives an interesting example of a group with a finite presentation of 3-deficiency one, and we comment on Ershov’s finitely presented Golod-Shafarevich group with property (T).

This paper is mostly an extract, which was under the supervision of Jack Button, of my PhD thesis.

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1. Main Result

There is an amount of ambiguity in saying that a group is of $p$-deficiency one. Formally, the $p$-deficiency of a group is the supremum of $\text{def}_p(\langle X|R \rangle)$ over all presentations $\langle X|R \rangle$ of the group. Therefore it is theoretically possible for the $p$-deficiency of a group to be one in the limit, but with none of $\text{def}_p(\langle X|R \rangle)$ being equal to one.

We avoid this delicate situation by insisting that the group has a presentation of $p$-deficiency one. This is the convention that we adopt whenever we deal with $p$-deficiency one groups.

First, we note a result from [1].

**Theorem 2.** [1] Let $G$ be a group with presentation

$$\langle a_1, \ldots, a_n | 1 = w_1^{r_1} = \ldots = w_m^{r_m} \rangle$$

where each $w_j$ is a word in the $a_i$ and their inverses. Suppose that $H$ is a normal subgroup of $G$ of index $N < \infty$ and that for each $j$, $w_j^k \not\in H$ for $k = 1, \ldots, r_j - 1$. Then the rank of the abelianization of $H$ is at least

$$1 + N \left( n - 1 - \sum_i \frac{1}{r_i} \right).$$

We now prove the following.

**Theorem 3.** Let $\Gamma$ be a finitely presented group with a presentation of $p$-deficiency equal to one, for some prime $p$. Then $\Gamma$ has a finite index subgroup $H$ that surjects onto $\mathbb{Z}$.

**Proof.** Let $\Gamma$ be a finitely presented group with a presentation of $p$-deficiency equal to one. So we have

$$\Gamma \cong \langle x_1, \ldots, x_d | w_1, \ldots, w_r, w_{r+1}^{a_{r+1}}, \ldots, w_q^{a_q} \rangle$$

with

$$\text{def}_p(\Gamma) = d - r - \sum_{i=r+1}^{q} \frac{1}{p^{a_i}} = 1$$

where $d, q \in \mathbb{N}$, $0 \leq r \leq q$, and $a_{r+1} \leq \ldots \leq a_q$ are positive integers.

For $i \in \{r + 1, \ldots, q\}$, we say that $w_i$ has exact order if there is some normal subgroup $H$ of finite index in $\Gamma$ such that $w_i^{p^{a_i}} \not\in H$. That is, $w_i$ has order $p^{a_i}$ as in the presentation above.

Consider $w_q$. Either $w_q$ has exact order in some finite index normal subgroup or it does not.

If it does not, consider now

$$G = \langle x_1, \ldots, x_d | w_1, \ldots, w_r, w_{r+1}^{p^{a_{r+1}}}, \ldots, w_{q-1}^{p^{a_{q-1}}}, w_q^{p^{a_q+1}} \rangle.$$
Note that
\[ \Gamma \cong G/\langle \langle w_q^n \rangle \rangle \]
and as \( \def_p(G) > 1 \) we have that \( G \) is \( p \)-large.

Recall the definition of \( p \)-large. Let \( H \) be a normal subgroup in \( G \) of index \( p^k \), for \( k \geq 0 \), such that there exists a surjection \( \psi : H \to F_2 \).

For ease of notation, we write \( w := w_q \) and \( a := a_q \). We may assume that the order of \( w \) in \( G \) is \( p^{a+1} \), as if \( o(w) < p^{a+1} \), then \( \Gamma = G \) is \( p \)-large, and we are done.

Now, our plan is to consider \( G/\langle \langle w^a \rangle \rangle \) and show that this quotient group has a finite index subgroup that surjects onto \( \mathbb{Z} \). As \( \Gamma \cong G/\langle \langle w^a \rangle \rangle \), this completes the proof of our theorem.

Consider the order of \( \overline{w} \), the image of \( w \) in \( G/H \).

a) If \( o(\overline{w}) \) in \( G/H \) is \( < p^a \), then this implies that \( w^{p^a-1} \in H \). We will show that \( G/\langle \langle w^a \rangle \rangle \) is \( p \)-large to obtain our result.

Let \( k_1, \ldots, k_s \) be a set of representatives for the cosets of \( H \) in \( G \). Let \( m \leq p^{a-1} \) be the smallest positive integer such that \( k_j w^m k_j^{-1} \in H \), for each \( j \). Note that \( m \) divides \( p^k = [G : H] \). Let \( n \) be any positive integer, and let \( G_{mn} = \langle \langle w^{mn} \rangle \rangle \) be the subgroup of \( G \) generated normally by \( w^{mn} \). Note that this is contained in \( H \), and is in fact the subgroup of \( H \) normally generated by \( \{ k_j w^m k_j^{-1} : 1 \leq j \leq s \} \). Now \( \{ \psi(k_j w^m k_j^{-1}) : 1 \leq j \leq s \} \) is a collection of elements in \( F_2 \). The key thing to note here, is that \( k_j w^m k_j^{-1} \) all have orders a power of \( p \) in \( H \), and so their images under \( \psi \) must be trivial in \( F_2 \). So \( G_{mn} \leq \ker \psi \), and we have the induced surjection \( \overline{\psi} : H/G_{mn} \to F_2 \). Now \( H/G_{mn} \) has finite \( p^i \)th power index in \( G/G_{mn} \). Therefore \( G/G_{mn} \) is \( p \)-large. Finally, we take \( mn = p^a \), and therefore \( G/\langle \langle w^a \rangle \rangle \cong \Gamma \) is \( p \)-large. The result now follows for \( \Gamma \).

b) If \( o(\overline{w}) \) in \( G/H \) is \( \geq p^a \), then in \( G/H/\langle \langle w^a \rangle \rangle \) the image of \( w \) has order dividing \( p^a \). As this is a finite \( p \)-quotient of \( \Gamma \), we use the fact that \( w \) has exact order \( p^a \) in \( G/\langle \langle w^a \rangle \rangle \), to deduce that \( w \) has exact order \( p^a \) in \( \Gamma \). As we are assuming here that \( w \) does not have exact order in \( \Gamma \), this case (b) is not possible.

So we see from the above that if \( w_q \) does not have exact order, then \( \Gamma \) is \( p \)-large, and the statement of the theorem is true.

Now we assume that \( w_q \) has exact order with respect to some finite index normal subgroup \( H_q \). Henceforth we only consider normal subgroups of \( \Gamma \) that are contained in \( H_q \).

Consider \( w_{q-1} \). Either \( w_{q-1} \) has exact order with respect to some finite index normal subgroup contained in \( H_q \), or it does not. If \( w_{q-1} \) does not have exact order, then similar to the above arguments, we have that \( \Gamma \) is \( p \)-large.

If \( w_{q-1} \) has exact order with respect to some finite index normal subgroup \( H_{q-1} \) contained in \( H_q \), then we henceforth only consider normal subgroups of \( \Gamma \) that are contained in \( H_{q-1} \).

And so on. Either \( \Gamma \) is proved to be \( p \)-large at some stage, or we end up with some finite index normal subgroup \( H_{r+1} \) such that \( w_k \) has exact order with respect to \( H_{r+1} \) for all \( k = r + 1, \ldots, q \). Then the rank of the abelianization of \( H_{r+1} \) is at least one by Theorem 2. Hence \( H_{r+1} \) surjects onto \( \mathbb{Z} \), as required. \( \Box \)
The second line of part (b) draws on the following simple fact from finite $p$-groups.

**Lemma.** Let $g$ be an element of a finite $p$-group $G$, and say $o(g) = p^k$ for $k > 0$. Let $N = \langle \langle g^{p^{k-1}} \rangle \rangle$. Then $g^{p^{k-2}} \notin N$.

**Proof.** We consider the Frattini subgroup $\Phi(N)$ of $N$, which is defined to be the intersection of all maximal subgroups of $N$. It is well-known that $\Phi(N)$ is characteristic in $N$, and that $\Phi(N) = N^{p}N'$ as $N$ is a finite $p$-group. As $\Phi(N)$ is characteristic in $N$, and $N$ is normal in $G$, we have that $\Phi(N)$ is normal in $G$.

Firstly, we note that we cannot have $g^{p^{k-1}}$ belonging to $\Phi(N)$: if $g^{p^{k-1}} \in \Phi(N)$, then all conjugates $h^{-1}g^{p^{k-1}}h$, for $h \in G$, also lie in $\Phi(N)$. This means that $N = \Phi(N)$, which is impossible.

Now suppose that $g^{p^{k-2}} \in N$. Then

$$(g^{p^{k-2}})^p = g^{p^{k-1}} \in N^p \leq \Phi(N),$$

a contradiction. Thus $g^{p^{k-2}} \notin N$, as required. \qed

**Remark 4.** The referee has provided a very nice alternative for part (a) of the proof of Theorem 3, as seen here.

Suppose that the $w_i$’s do not have exact order. Then for every finite index normal subgroup $H$ of $\Gamma$ there exists some $k$ such that $w_k^{p^{k-1}} \in H$. So under the application of the Reidemeister-Schreier rewriting process, the relators involving $w_k$ remain $p^{th}$ powers in the presentation for $H$. Thus, on computing the rank of the mod $p$ homology of $H$ (i.e. $d_p(H)$), we may disregard contributions from these relators involving $w_k$.

Applying this to the derived $p$-series of $\Gamma$ (i.e., $\Gamma^{(0)} = \Gamma$, $\Gamma^{(i)} = [\Gamma^{(i-1)}, \Gamma^{(i-1)}][\Gamma^{(i-1)}]^p$ for $i \in \mathbb{N}$), we deduce that the derived $p$-series of $\Gamma$ has linear growth of mod $p$ homology. Theorem 1.12 of [10] now implies that $\Gamma$ is $p$-large.

2. **Corollaries**

Theorem 3 together with Corollary 2.4 of [5] imply the following response to the Related Burnside Problem.

**Corollary 5.** Let $G$ be an infinite finitely presented group with a presentation of $p$-deficiency greater than or equal to one, for some prime $p$. Then $G$ is not torsion.

The next corollary incorporates Kazhdan’s property (T).

**Corollary 6.** Let $G$ be an infinite finitely presented group with a presentation of $p$-deficiency greater than or equal to one, for some prime $p$. Then $G$ does not have property (T).

**Proof.** By Theorems 1 and 3, we know that $G$ has a finite index subgroup $H$ such that $H$ surjects onto $\mathbb{Z}$. The result now follows from Corollary 1.3.6 and Corollary 1.7.2 of [3]. \qed
Next, we have the following result from [9].

**Theorem 7.** [9] Let $G$ be a finitely generated, large group and let $g_1, \ldots, g_r$ be a collection of elements of $G$. Then for infinitely many integers $n$, $G/\langle\langle g_1^n, \ldots, g_r^n \rangle\rangle$ is also large. Indeed, this is true when $n$ is any sufficiently large multiple of $[G : H]$, where $H$ is any finite index normal subgroup of $G$ that admits a surjective homomorphism onto a non-abelian free group.

Part (a) of the proof of Theorem 3 follows the proof of Theorem 7 closely. Below is a stronger statement for free groups which is used in the proof of Theorem 7.

**Theorem 8.** [9] Let $F$ be a finitely generated, non-abelian free group. Let $g_1, \ldots, g_r$ be a collection of elements of $F$. Then, for all but finitely many integers $n$, the quotient $F/\langle\langle g_1^n, \ldots, g_r^n \rangle\rangle$ is large.

The above theorem has a topological proof. As in the proof of the Nielsen-Schreier Theorem on subgroups of free groups, $F$ here is viewed as the fundamental group of a bouquet of circles. Then the quotient $F/\langle\langle g_1^n, \ldots, g_r^n \rangle\rangle$ is obtained by attaching 2-cells representing $g_1^n, \ldots, g_r^n$ along the circles. More details are to be found in [9].

Olshanskii and Osin give a shorter algebraic proof of Theorem 7 in [11]. The main body of Olshanskii and Osin’s proof relies on Theorem 8, which they also prove with alternative algebraic arguments.

We remark here that Theorem 8 (and hence Theorem 7) follows from Theorem 1. We remind the reader that Theorem 1 relies on another result of Lackenby ([10], Theorem 1.15).

**Corollary 9.** Let $F$ be a free group of rank $r \geq 2$, with $g_1, \ldots, g_k$ arbitrary elements of $F$. Then $\overline{F} \cong F/\langle\langle g_1^n, \ldots, g_k^n \rangle\rangle$ is large for all but finitely many $q \in \mathbb{N}$.

**Proof.** We consider the $p$-deficiency of $\overline{F}$:

$$\text{def}_p(\overline{F}) \geq r - \frac{k}{p^{l_p}},$$

where $p$ is some prime factor of $q$, and $l_p$ is the highest power of $p$ dividing $q$. By Theorem 1, the group $\overline{F}$ is $p$-large if $\text{def}_p(\overline{F}) > 1$, that is, when $p^{l_p} > \frac{k}{r-1}$.

So as long as $p^{l_p} > \frac{k}{r-1}$ for at least one $p$ dividing $q$, then $\overline{F}$ is large. That is, for all but finitely many $q \in \mathbb{N}$, the group $\overline{F}$ is large. □

Lackenby’s proof of Theorem 8 (or Corollary 9) relies on topological arguments, which span over a few pages. Here, Theorem 1 has enabled us to present a short proof of a different spirit.
3. Examples

Clearly finitely presented groups of $p$-deficiency one exist and examples include the infinite dihedral group $D_\infty = \langle x_1, x_2 | x_1^2, x_2^2 \rangle$, all groups of deficiency one, and the group $P = \langle x, y, z | x^3, y^3, z^3, (xy)^3, (xz)^3, (yz)^3 \rangle$. The group $D_\infty$ is not torsion nor large. The groups of deficiency one are not torsion but some are large (see [4]). The group $P$ was verified by MAGMA to be 3-large (and hence is not torsion). For the group $P$, we used the approach shown below, as is similar to Subsection 4.2 of [5].

Claim. The group $P = \langle x, y, z | x^3, y^3, z^3, (xy)^3, (xz)^3, (yz)^3 \rangle$ is 3-large.

Proof. Using MAGMA’s LowIndexNormalSubgroups function, we considered the following index three normal subgroup of $P$:

$C = \langle a, b, c, d | [c^{-1}, a^{-1}], [d^{-1}, b^{-1}], adc^{-1}a^{-1}bcd^{-1}b^{-1} \rangle,$

which was (at the time of writing) seventh on the list of fourteen normal subgroups with index at most three in $P$. The above presentation for $C$ was obtained using MAGMA’s Simplify function.

Then we formed the quotient

$C/\langle c,d \rangle$

and we noticed that the quotient is isomorphic to $\langle a, b \rangle \cong F_2$. Hence $C$ is 3-large by definition, and since $C$ is normal in $P$ of index 3, we have proved that $P$ is 3-large, as required.

With reference to Corollary 6, the following example is due to Ershov & Jaikin-Zapirain ([7], Proposition 7.4). Let $d \geq 6$ and $p > (d - 1)^2$, then the group

$G \cong \langle x_1, \ldots, x_d | [x_i, x_j], x_i^p = 1 \forall i \neq j, x_i^p = 1 \rangle$

is a finitely presented Golod-Shafarevich group with property (T) (see [6] for further information). Naturally the $p$-deficiency of $G$ is not one.

References


Magdalene College, Cambridge, CB3 0AG, United Kingdom.
E-mail address: anitha.t@cantab.net